Illustration of GA using GABLE

SIGGRAPH 2001, Course #53

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DEMOvectors

Geometric Algebra

- The geometric product ab does it all
- Algebraically, it is
  - linear
  - associative
  - non-commutative
  - invertible
- We will visualize these properties

Properties

Geometry	Algebra
$a \wedge b$ spanning	anti-commutation $\frac{1}{2}(ab - ba)$
$a \cdot b$ complementation perpendicularity	commutation $\frac{1}{2}(ab + ba)$
orthogonalization	invertibility
rotation	exponentiation

Derived products

- $x \cdot a$ = symmetric part of $x\,a$

$$x \cdot a \equiv \frac{1}{2}(xa + ax)$$

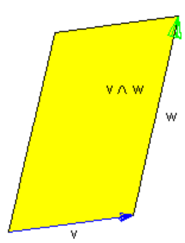
- $x \wedge a$ = anti-symmetric part of $x\,a$

$$x \wedge a \equiv \frac{1}{2}(xa - ax)$$

- Decomposition of geometric product

$$xa = x \cdot a + x \wedge a$$

Outer product: spanning



$$a \wedge b = -b \wedge a$$

- dimensionality
- attitude
- sense
- magnitude

DEMOouter

- Given a, all x with same $x \wedge a$ are on a line
- Extension: $a \wedge b \wedge c$ is a volume
- Vectors, bivectors, trivectors, etc.
  All elements of geometric algebra
- $\dim(A \wedge B) = \dim(A) + \dim(B)$
  (but beware of overlap)

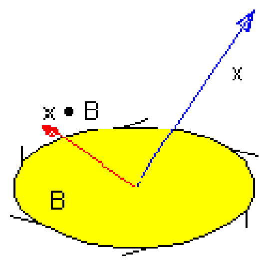
Inner product: perpendicularity

$$a \cdot b = b \cdot a$$

- $A \cdot B$ is part of B perpendicular to A

  DEMOinner

- Given a, all x with same $x \cdot a$ are on a hyperplane



- $\dim(A \cdot B) = \dim(B) - \dim(A)$

Parallel Component

Consider $x = x_\perp + x_{||}$ relative to some vector a

- Geometrically: $x_{||}$ is part of x parallel to a
- Classically: $x_{||} \cdot a = x \cdot a$ and $x_{||} \wedge a = 0$
- Geometric Algebra: add them and divide

$$x_{||}a = x_{||} \cdot a + x_{||} \wedge a = x_{||} \cdot a = x \cdot a$$

  Solvable: $x_{||} = (x \cdot a)/a$

Perpendicular Component

- Geometrically: $x_\perp$ is part of x perpendicular to a
- Classically: $x_\perp \wedge a = x \wedge a$ and $x_\perp \cdot a = 0$
- Geometric Algebra: $x_\perp a = x \wedge a$

  Solvable: $x_\perp = (x \wedge a)/a$

DEMOproj

Geometric Product is Invertible

- $xa = x \cdot a + x \wedge a$ is invertible

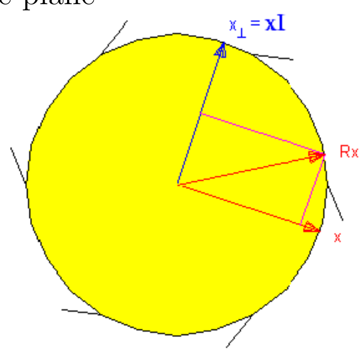
  DEMOinvertible

$$x = (xa)/a = (x \cdot a)/a + (x \wedge a)/a$$

- Can divide by vectors, bivectors

Rotations

- Many ways to do rotations in geometric algebra
- Given x and plane I containing x (so $x \wedge I = 0$)
  Rotate x in the plane



- Coordinate free view

  $\mathsf{R}x$ = bit of x and bit of perpendicular to x

  (amounts depend on rotation angle)
- Perpendicular to x in I plane (anti-clockwise) is

$$x \cdot I = xI = -Ix$$

  DEMOrotdefinition

- Rotation as post-multiply:

$$\mathsf{R}x = x(\cos\phi) + (xI)(\sin\phi) = x(\cos\phi + I\sin\phi)$$

- Rotation as pre-multiply:

$$\mathsf{R}x = (\cos\phi) + (\sin\phi)(-Ix) = (\cos\phi - I\sin\phi)x$$

Complex Rotations

- Related to complex numbers
$$II = -1$$
but I has a geometrical meaning since $xI = -Ix$
- We can write $\boxed{\cos\phi + I\sin\phi = e^{I\phi}}$
- Each rotation plane has own bivector I
so many "complex numbers" in space
- Bivector basis ($\mathbf{i} = e_2 \wedge e_3$, $\mathbf{j} = e_3 \wedge e_1$, $\mathbf{k} = e_1 \wedge e_2$)
$$I = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$$

Rotations in 3D

- Pick rotation plane I and (possibly non-coplanar) vector x
$$x = x_\perp + x_{||}$$
Would like to get $\mathsf{R}_{I\phi}x = x_\perp + \mathsf{R}_{I\phi}x_{||}$.
- $x_{||}$ rotation:
either $e^{-I\phi}x_{||}$ or $x_{||}e^{I\phi}$ (or even $e^{-I\phi/2}x_{||}e^{I\phi/2}$)
- $x_\perp$ rotation:
$$x_\perp e^{I\phi} = \underbrace{\cos\phi\, x_\perp}_{\textit{vector}} + \underbrace{\sin\phi\,(x_\perp I)}_{\textit{trivector}}$$
$$e^{-I\phi}x_\perp = \cos\phi\, x_\perp - \sin\phi\,(Ix_\perp)$$
$$\begin{aligned}
(e^{-I\phi}x_\perp)e^{I\phi} &= \cos\phi\, x_\perp e^{I\phi} - \sin\phi\, Ix_\perp e^{I\phi} \\
&= \cos^2\phi\, x_\perp + \cos\phi\sin\phi\, x_\perp I \\
&\quad - \sin\phi\cos\phi\, Ix_\perp - \sin^2\phi\, Ix_\perp I \\
&= \cos^2\phi\, x_\perp - sin^2\phi\, IIx_\perp \\
&= (\cos^2\phi + \sin^2\phi)x_\perp \\
&= x_\perp
\end{aligned}$$
- Bottom line:
$$e^{-I\phi/2}xe^{I\phi/2} = x_\perp + \mathsf{R}_{I\phi}x_{||} = \mathsf{R}_{I\phi}x$$

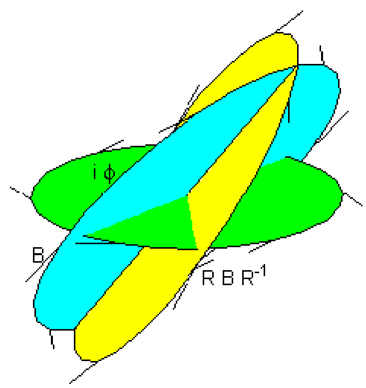
Rotors

DEMOrotor

- So $\mathsf{R}_{-I\phi}x = e^{-I\phi/2} x e^{I\phi/2}$
- Further,

$$\boxed{\mathsf{R}_{-I\phi}X = e^{-I\phi/2} X e^{I\phi/2} = RXR^{-1}}$$

  where X is any geometric object (vector, plane, volume, etc.)



- $R = e^{-I\phi/2}$ is called a *rotor*

  $R^{-1} = e^{I\phi/2}$ is called the *inverse rotor*

Quaternions

- A rotor is a (unit) quaternion
- **i**, **j**, **k** are not complex numbers, they are
  - bivectors (not vectors!)
  - rotation operators for the coordinate planes
  - basis for planes of rotation
  - an intrinsic part of the algebra

Composing Rotations

Composition of rotations through multiplication

$$(\mathsf{R}_2 \circ \mathsf{R}_1)x = R_2(R_1 x R_1^{-1})R_2^{-1} = (R_2 R_1)x(R_2 R_1)^{-1}$$

- $R_2 R_1$ is again a rotor.
  It represents the rotation $\mathsf{R}_2 \circ \mathsf{R}_1$
- Note: use geometric product to multiply rotors/quaternions
  No new product is needed

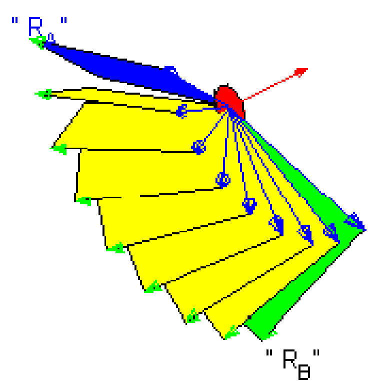
Interpolation

From rotor R_A to rotor R_B in n similar steps:

$$R^n R_A = R_B \quad \Longleftrightarrow \quad R = (R_B/R_A)^{1/n}$$

So

$$R = (e^{I\phi/2})^{1/n} = e^{I\phi/(2n)}$$



DEMOinterpolation

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All you need is blades

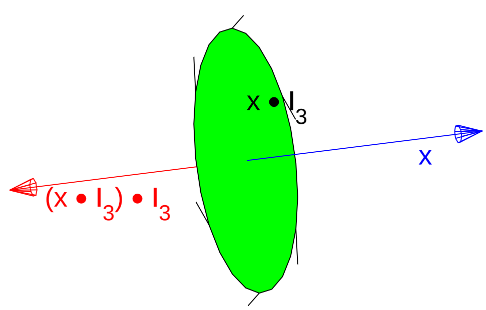
- *'Vector space model'*: k-blades (made by '$\wedge$') are quantitative oriented k-dimensional subspace elements
- But we would like to represent 'offset' subspaces.
- This leads to the *affine model* (for flat subspaces) and to the *homogeneous model* (spheres as subspaces).

Dualization

- $\mathbf{I}_m$ is the *pseudoscalar* of m-space (highest order blade, volume element)
- A^* is part of $\mathbf{I}_m$-space perpendicular to A:

$$A^* \equiv A \cdot \mathbf{I}_m$$

- Example: bivector $\mathbf{B}$, then $\mathbf{B}^* = -\mathbf{n}$, normal vector



DEMOdual

Cross product and normal vectors

- Cross product in 3D dual of outer product:

$$a \times b \equiv -(a \wedge b) \cdot \mathbf{I}_3$$

- Under a linear transformation f

$$\begin{aligned} f(a \times b) &= \bar{f}^{-1}(a) \times \bar{f}^{-1}(b) \det f \\ f(a \wedge b) &= f(a) \wedge f(b) \end{aligned}$$

- Use $\wedge$ instead of $\times$

Meet

- Intersection operation is 'dual of spanning' in their common space: $(A \cap B)^* = B^* \wedge A^*$. This gives

$$A \cap B = B^* \cdot A$$

- This is called the meet of A and B.

  DEMOmeetplanes

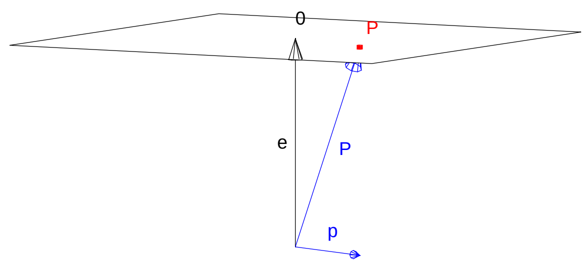
- Well-known special case: meet of two planes in $\mathbf{I}_3$,

$$\mathbf{A} \cap \mathbf{B} = \mathbf{B}^* \cdot \mathbf{A} = \mathbf{A}^* \times \mathbf{B}^* = \mathbf{n_A} \times \mathbf{n_B}$$

  but above formula applies to *any* intersection.

Affine model

- The framework for 'homogeneous coordinates' and 'Plücker coordinates'
- Get affine/homogeneous spaces by using one dimension for "point at zero"
  - **Point:** $P = e + \mathbf{p}$ such that $e \cdot \mathbf{p} = 0$
  - **Vector:** $\mathbf{v}$ such that $e \cdot \mathbf{v} = 0$
  - **Tangent plane:** bivector $\mathbf{B}$ such that $e \cdot \mathbf{B} = 0$



DEMOaffine

Affine representation

- **Line:** point P, point Q

$$L = P \wedge Q = (e + \mathbf{p}) \wedge (e + \mathbf{q}) = e \wedge (\mathbf{q} - \mathbf{p}) + (\mathbf{p} \wedge \mathbf{q})$$

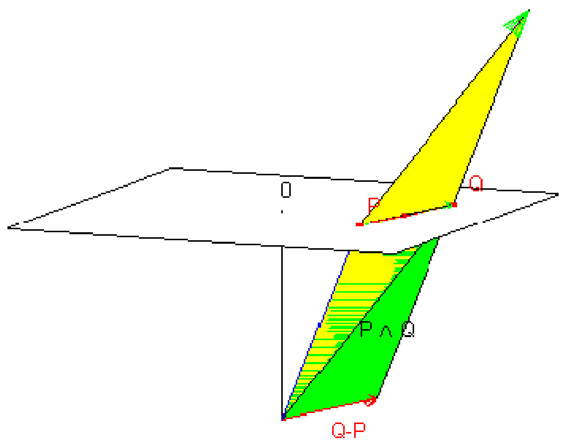
- **Line:** direction $\mathbf{v}$, point P

$$L = P \wedge \mathbf{v} = e\,\mathbf{v} + \mathbf{p} \wedge \mathbf{v}$$

- **Plane:** '2-direction' bivector $\mathbf{B}$, point P

$$\Pi = P \wedge \mathbf{B} = e\,\mathbf{B} + \mathbf{p} \wedge \mathbf{B}$$

Composite objects: use '$\wedge$', '$\cdot$', '$\cap$' and dual.



Plücker Revisited

	GA	Plücker
point	$\mathbf{p} + e$	$(\mathbf{p}, 1)$
line	$e \wedge (\mathbf{q} - \mathbf{p}) + \mathbf{p} \wedge \mathbf{q}$ $= (\mathbf{p} - \mathbf{q})e + (\mathbf{p} \times \mathbf{q})\mathbf{I}_3$	$(\mathbf{p} - \mathbf{q}, \mathbf{p} \times \mathbf{q})$
plane	$e\,\mathbf{B} + \mathbf{p} \wedge \mathbf{B}$	?
dual plane	$\mathbf{B}^* - (\mathbf{p} \cdot \mathbf{B}^*)e$ $= -(\mathbf{n} - (\mathbf{p} \cdot \mathbf{n})e)$	$[\mathbf{n}, -\mathbf{p} \cdot \mathbf{n}]$

GA 'labels' 1, e and $\mathbf{I}_3$ determine multiplication and interpretation rules automatically

Affine representation: examples

- *Example 1:* Intersection of line $L = \mathbf{u}e + \mathbf{v}\mathbf{I}_3$ and (dual) plane $\Pi^* = \mathbf{n} - \delta e$ is:

  $$\Pi \cap L = \Pi^* \cdot L = -(\mathbf{n} \cdot \mathbf{u})e - (\mathbf{v} \times \mathbf{n} - \delta \mathbf{u})$$

  The 'labels' tell us that this is a *point* at location:

  $$\frac{\mathbf{v} \times \mathbf{n} - \delta \mathbf{u}}{\mathbf{n} \cdot \mathbf{u}}$$

- *Example 2:* Distance of point P to plane Π^*:

  $$\Pi \cap P = \Pi^* \cdot P = \delta - \mathbf{n} \cdot \mathbf{p}$$

  Scalar outcome: oriented distance.

- *Example 3:* Intersecting lines DEMOaffinemeet

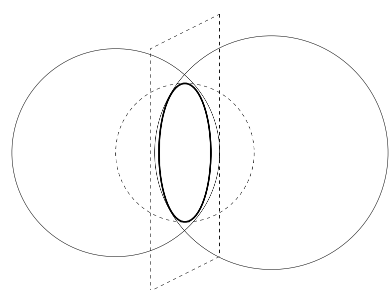
Homogeneous Model

- Points are vectors p, q
- Distances directly as $p \cdot q = -\frac{1}{2}(\mathbf{p} - \mathbf{q})^2$
- Special point at infinity e_∞: $(e_\infty)^2 = 0$, $e_\infty \cdot p = 1$
- Altogether $(m+2)$-space representing E^m
- Blades represent k-spheres: 3-sphere $p \wedge q \wedge r \wedge s$
- Flats are spheres through infinity: line $e_\infty \wedge p \wedge q$
- Very compact intersections, reflections, etc.

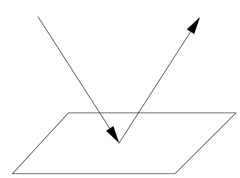
Spheres and planes

- Sphere (c, ρ) is dually the vector $\sigma = c + \frac{1}{2}\rho^2 e_\infty$
- Plane $(\mathbf{n}, \delta)$ is $\pi = \mathbf{n} - \delta e_\infty$
- Sphere σ perpendicular to plane π obeys $\pi \cdot \sigma = 0$.
- Intersect two spheres:

$$\sigma_1 \wedge \sigma_2 = \underbrace{\frac{\sigma_1 \wedge \sigma_2}{\sigma_2 - \sigma_1}}_{\textit{perp. sphere}} \wedge \underbrace{(\sigma_2 - \sigma_1)}_{\textit{int. plane}}$$



- Reflect line ℓ in plane π: $-\pi\,\ell\,\pi$.



Computational issues

- Actual geometrical computations like Plücker coordinates, so rather efficient.
- However, potential basis for elements much bigger: 2^{n+2} for homogeneous model of n-space (i.e. 32 for 3-space).
- All products are *linear*, so expressible as matrix multiply: $a \wedge b \quad \rightarrow \quad [a^\wedge][b]$, for 32×32 matrices. Some reducing tricks possible (and so done in GABLE), but too expensive in time and space.
- Should make efficient coding of only the necessary elements involved in a computation. Gives Plücker efficiency for spheres.

GABLE is freeware

For a free copy of GABLE and a geometric algebra tutorial, see
`http://www.science.uva.nl/~leo/clifford/gable.html`
`http://www.cgl.uwaterloo.ca/~smann/GABLE/`