# Chapter 5

# CONCLUSIONS AND FUTURE WORK

This dissertation has aimed to explore the possibility of using modern mathematics and computer science to develop new ways of analyzing and synthesizing ornament. Although I chose to focus specifically on Islamic star patterns and Escher's tilings, the underlying goal was to develop general ideas and principles that might then be applied to other ornamental styles. The range of possible styles to investigate is huge, as evidenced by a quick look through a treasure trove like *The Grammar of Ornament* [91]. We should also consider moving on with new forms of ornament that are inseparably tied to the computer. What new vistas in ornamentation will be opened up by the increasing power of computer graphics?

What follows are some further ideas for future work that are not directly related to the work of Chapters 3 and 4.

## *5.1 Conventionalization*

Conventionalization refers to the creation of stylized, iconic interpretations of natural forms, most commonly plants. There is a general trend in the ornamental traditions of many cultures for conventional representations to evolve over time. Wong *et al.* point out that one goal of conventionalization is to distill a naturalistic form down to an abstract essence, freed from the idiosyncracies of any specific instance of the form [139]. Another reason for simplified designs might simply be the practicality of working with certain materials.

Over time, a conventionalized form begins to take on a life of its own, forming its own visual language that evolves as it is passed through history from designer to designer. Gombrich, writing on "The Etymology of Motifs" [60, Page 180], discusses some examples of this progression, including conventionalized representations of the lotus and the acanthus. In another fascinating example, Christie shows how many medieval European frieze patterns evolved from conventionalization of

Arabic calligraphic inscriptions on earlier silk weavings [22, p. 20].

Conventionalization plays an important role in Islamic star patterns. A star pattern is not always depicted as an isolated arrangement of geometric forms. Frequently, the polygonal regions defined by the lines of a star pattern are filled not with solid colours, but with elegant floral designs that extend more or less to the boundary of the region. The wonderful drawings of Prisse d'Avennes, recently reprinted [31], illustrate many examples. The designs are closely associated with arabesques, another form of Islamic ornamentation. Typically, each distinct shape of region has a floral motif assigned to it, and the motif has the same symmetries as its surrounding region. It should be possible to extend the work of Wong *et al.* [139] to create symmetric ornament inside of symmetric boundaries, and to then use this extension to produce appropriate floral motifs for every distinct cell shape in a given star pattern.

Escher's tilings also rely heavily on conventionalization, this time of animal forms. Nobody would mistake the outlines in Escher's sketchbook for real-world animals. Each form is a cartoon, a highly stylized interpretation that nevertheless is highly suggestive of an animal. In some cases, conventionalization gives way to outright invention: shapes are decorated with suggestive eyes and appendages, but are not meant to depict any real animal. Escher uses stylization to great advantage. In freeing his cartoon animals from constraints of realism, he can distort them into shapes that tile the plane without sacrificing the "meaning" of the finished design. His success suggests another direction for future work in conventionalization. To aid the Escherization algorithm, we might turn real-world outlines into conventionalized representations endowed with new degrees of freedom. By varying the degrees of freedom, we might locate a version of the shape that lends itself more readily to tiling the plane.

The long-term goal in this direction would be to devise an algorithm that produces conventionalizations from real-world objects (images or three dimensional models) without any guidance. I believe we are still a long way from fully automatic conventionalization. Since abstraction relies on true understanding of the object being abstracted, an automated process would seem to require real machine intelligence. A feasible intermediate goal would be to investigate what sorts of high level tools might be uniquely qualified to aid the user in the process of conventionalization. Recently, Santella and DeCarlo demonstrated a system that evaluates salience in an image based on eye-tracking data from a human user [32]. They use the measure of salience to guide a painterly rendering algorithm, but perhaps eye-tracking data could also be used to determine "geometric salience," a measure of the relevance of parts of an object's outline.

### *5.2 Dirty symmetry*

Glassner argues that too much order can be just as unappealing as not enough [54]. A floor covered by a grid of square tiles is so featureless that perhaps it ought not be considered ornamental at all.

We have already seen ways to tamper with the global order of symmetry. Rigid motions are lost in translation from the hyperbolic plane to the Poincaré model, but the projected design still has order. Expanding the horizons of symmetry to include quasiperiodic tilings or fractals questions but ultimately reaffirms our perception of order.

The perfection of symmetry can be tampered with even more easily. For example, the entire plane can be passed through a displacement field based on procedural noise [44]. Motifs in a pattern are distorted slightly, obliterating every symmetry. However, when the magnitude of the displacement is not too great, we have no difficulty whatsoever seeing both the symmetric "essence" of the pattern and the concrete deviation from that essence. The new pattern has the best of both worlds, conveying an organic, loose appearance while hinting at a rigid underlying structure. Møller and Swaddle cite psychoaesthetic and sociological evidence that this sort of imperfect symmetry is preferred over perfection [113].

The preference is well explained by Gombrich's "sense of order" [60]. He adopts an outlook on perception that is very much in line with information theory. We perceive structure in the world by first forming a mental model that predicts perfect regularity, and then evaluating the perceived deviation from this model. We can theorize that a randomly displaced pattern stimulates and engages the perceptual system at both the model-forming and the deviation-measuring levels. Gombrich claims that when the mental model completely explains what is actually seen, the act of perception is made too easy and the result is boredom. On the other hand, a completely irregular pattern allows for no model; robbed of the ability to make sense of such a pattern, the result is confusion. As Gombrich says, delight lies somewhere in between these two extremes [60, Page 9].

It would be interesting to explore how this "dirty symmetry" affects aesthetic judgment of a pattern, in terms of the variety and magnitude of the distortion. Computers are ideally suited to the task, allowing us to perform random distortions with ease. As Gombrich says, “Maybe the greatest novelty here is the ability of computers not only to follow any complex rule of organization but also to introduce an exactly calculated dose of randomness [60, Page 94].” We could use these “calculated doses of randomness” to measure experimentally the aesthetic response to varying amounts of distortion. In image processing, a small amount of colour noise can improve the qualitative appearance of an image. Can geometric noise be added to any design to increase its appeal?

### *5.3 Snakes*

In Section 4.7, I discussed Escher’s lifelong quest to capture and represent infinity. Through his own investigations, he discovered methods to make motifs diminish in size towards the center of a disk. Later he found in Coxeter’s writings on non-Euclidean geometry the key to putting the limit on the outer edge of the disk, thus capturing an entire universe on a finite page.

In his final print, *Snakes*, Escher fuses these two ideas with astonishing mathematical ingenuity. *Snakes* features an arrangement of interlocking rings that diminish in size both at the center and at the outer edge of a disk. Here we have the infinities of both sorts of limiting patterns, still confined to a finite space with no possibility for further extension.

Very little has been written about the geometry of *Snakes*. Ernst [47, Page 110] acknowledges that the “Coxeter network” (a hyperbolic tesselation in the Poincaré model) describes the outer rings. Ernst’s description is also valuable because it includes several of Escher’s many preliminary sketches for this print; many more sketches can be found in a wonderful new catalog of Escher’s work [105]. Rigby [122] goes into much more detail in describing the outer and inner rings.

In some of Escher’s preliminary sketches, we see a network of triangles made by drawing line segments joining the centers of adjacent rings. Let us consider instead the dual of this network, which can be constructed by joining the points where triplets of rings pass by each other in a weave. This network will be a tiling where each tile corresponds to one ring. Rigby shows that in the outer portion of the design, the tiling is just the hyperbolic $(6.8^2)$, which can be constructed by truncating the regular tiling $(4^6)$.

Ernst does not attempt to describe the structure of the rings at the center of *Snakes*. Rigby describes the structure as a “radiating framework,” but does not give details on how such a framework

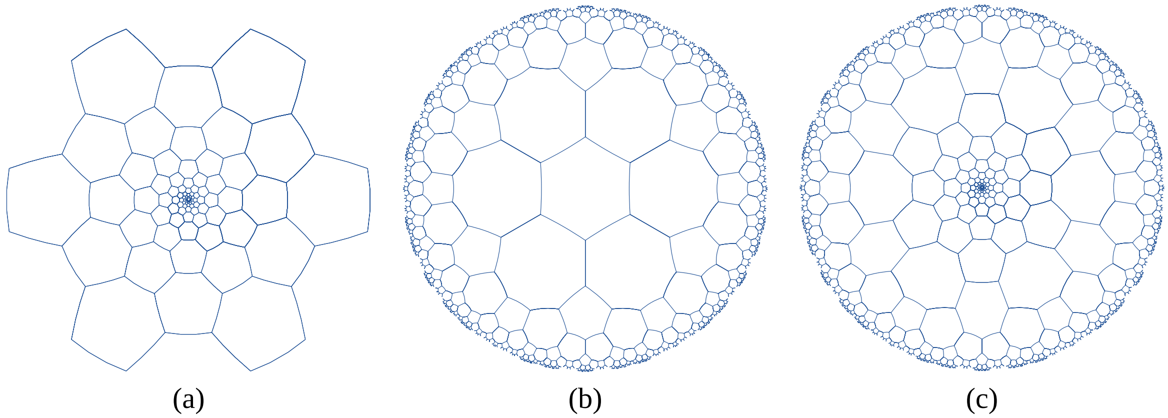



**Figure 5.1** A visualization of the geometric basis of Escher's *Snakes*. In the three tilings, the tiles represent rings from the original print. The tiling in (a) corresponds to the inner portion of the design, where rings diminish in size towards the center of a disk. The tiling in (b) is the hyperbolic tiling corresponding to the outer part of *Snakes*. In (c), the two are roughly combined in a manner consistent with Escher's design.

may be constructed. The answer can be found in Dixon's "antiMercator" transformation [37], which can be defined in the complex plane by $f(z) = e^z$. The antiMercator operation transforms an infinite horizontal strip of width $2\pi$ to the whole plane. Horizontal lines are mapped to lines radiating out of the origin, and vertical lines to circles. Other lines are mapped to equiangular spirals.

Given any periodic tiling with minimal translation vectors $\overrightarrow{T_1}$ and $\overrightarrow{T_2}$, and any integers $a$ and $b$ (not both zero), we can scale and rotate the tiling so that the vector $a\overrightarrow{T_1} + b\overrightarrow{T_2}$ is vertical with length $2\pi$. The antiMercator transformation of the scaled and rotated tiling is an attractive pattern that tiles the plane with a single limit point (a place where infinitely many tiles meet) at the origin. In general, antiMercator transforms of periodic tilings resemble phyllotaxis patterns, with counterrotating spirals of tiles emanating from the origin. In the case of *Snakes*, the central arrangement of rings can now be viewed as the antiMercator transform of the regular tiling by hexagons with $a = 6$ and $b = 6$.

Of course, the trick lies not just in defining the outer hyperbolic and inner antiMercator tilings, but in stitching the two together in a hybrid. Here Escher's intuition guides him flawlessly. Rigby shows how Escher makes the two parts of the design link up by turning some of the circular rings into ovals. Examination of *Snakes* also reveals one set of heptagonal tiles (or rings that are linked to seven others) in the stitched-together region of the design. For comparison, the antiMercator, hyperbolic, and stitched-together tilings are shown in Figure 5.1. Some of the tiles in the crossover region are more distorted than in Escher's design, but the topology is correct relative to the original print.

Is there a general theory that can produce these marvelous hybrid tilings? It seems as if many pairs of antiMercator and hyperbolic tilings might be fused together to serve as a basis for new doubly-infinite patterns. The challenge is to find a general theory for determining which pairs are compatible, and for stitching the tilings together when they are. Aside from linked rings, we can imagine using these tilings as a basis for other sorts of ornament, such as Escher tilings or Islamic star patterns. I find this future direction particularly appealing, since *Snakes* is seemingly the only artwork ever created with this geometry.

*Snakes* is just one example of a generalization of planar symmetry that is still highly regular. Many other opportunities await for exploring similar kinds of hybrid structures and the ornamental patterns that can be derived from them.

## *5.4 Deformations and metamorphoses*

Many of Escher's prints feature divisions of the plane that change or evolve in some way. The most well-known is probably *Metamorphosis II*, a long narrow print containing a variety of ingenious transitions between patterns, tilings, and realistic scenery. Escher was quite explicit about the temporal aspect of these long prints. He would not simply describe the structure of *Metamorphosis II* – he would narrate it like a story [49, Page 48].

A survey of Escher's work (as collected by Bool *et al.* [15]) turns up sixteen pieces employing some kind of transition device. By studying these sixteen pieces, I have identified six categories of transition. *Metamorphosis II* serves as a kind of atlas, as it incorporates all six varieties. They are as follows:

T1. **Realization:** A geometric pattern is elaborated into a landscape or other concrete scene. In *Metamorphosis II*, a cube-like arrangement of rhombs evolves into a depiction of the Italian town of Atrani.

T2. **Interpolation:** A tiling evolves into another tiling by smoothly deforming the shapes of tiles. Escher used this device to change simple tilings into his familiar interlocking animal forms (for example, squares into reptiles in *Metamorphosis II*, and triangles into a variety of forms in *Verbum*).

T3. **Sky and Water:** Realistic shapes meet in a dihedral tiling. We have already encountered this sort of transition in Section 4.6.2, in the discussion of Sky and Water designs. This sort of transition starts with copies of some realistic shape $A$, ends in copies of another realistic shape $B$, and moves between them by passing through a dihedral tiling whose tiles resemble $A$ and $B$. This device is used to produce single transitions, as in *Sky and Water*, and also as part of longer structures, as in *Metamorphosis II* (the transitions in the latter are rotated ninety degrees with respect to the former).

T4. **Abutment:** Two distinct tilings are abruptly spliced together along a shared curve. The transition works when the two tilings have vaguely similar geometry and can be made to abut one another without too much distortion. Escher uses this device exactly once, to transition from hexagonal reptiles to square reptiles in *Metamorphosis II* (later, he embedded the same sequence into the larger *Metamorphosis III*).

T5. **Growth:** Motifs gradually grow to fill the negative space in a field of pre-existing motifs, resulting in a multihedral tiling. Often, after a Sky and Water transition, the result is a pattern of realistic motifs that do not tile. In several cases, Escher transitions back to tilings by growing another set of motifs into the empty spaces of the pattern. The new motifs need not occupy all the empty space; in *Metamorphosis II*, red birds grow to occupy half the space between black birds. When the two sets of motifs finally fit together, they leave behind a white area in the form of a third bird motif.

T6. **Crossfade:** Two designs with compatible symmetries are overlaid, with one fading into the other. Escher also applies this device sparingly, using it only to transition from a rectilinear arrangement of copies of the word “metamorphose” into a checkerboard (and later, to make the reverse transition).

Putting all these transition types together, the sequence of transitions in *Metamorphosis II* might then be read as

- **T6** (copies of "metamorphose" into a checkerboard)
- **T2** (a checkerboard into a square arrangement of reptiles)
- **T4** (square reptiles into hexagonal reptiles)
- **T2** (hexagonal reptiles into hexagons)
- **T1** (hexagons into a honeycomb with bees)
- **T3** (bees into fish)
- **T3** (fish into black birds)
- **T5** (black birds into birds of three different colours)
- **T2** (birds into a cube-like arrangement of rhombs)
- **T1** (rhombs into the town of Atrani, which then becomes a chessboard)
- **T1** (a chessboard into an orthographic checkerboard — an elaboration in reverse)
- **T6** (a checkerboard into copies of "metamorphose")

By far, the most important transition type in Escher's work is the Sky and Water device. We have seen how the Heaven and Hell Escherization algorithm of Section 4.6.2 might be applied to build Sky and Water designs. We might then consider that technique to be a first step in the creation of a "Metamorphosis toolkit," a system that would simplify the construction of images like Escher's. The other five transition types hold a collection of interesting challenges. Crossfade seems to be primarily a matter of registration and image processing. Growth seems approachable, though emulating Escher's use of growth would require a means of representing 3-isohedral tilings. Abutment is rather special, and would probably not be very widely applicable. Nevertheless, a program could examine paths that follow tiling edges in the two tilings, in search of two paths that resemble each other as much as possible. Elaboration seems like a fascinating but very difficult problem. Given a tiling and, say, an image, an elaboration algorithm would have to search the image for a region that could be expressed as a gradual geometric enrichment of the tiling. The image might also need to be distorted in a very specific way to accommodate the tiling.

Interpolation is a beautiful mathematical problem that deserves a more extended discussion. Given two tilings $\mathcal{T}_1$ and $\mathcal{T}_2$, we ask for a smooth geometric transition between the two tilings.

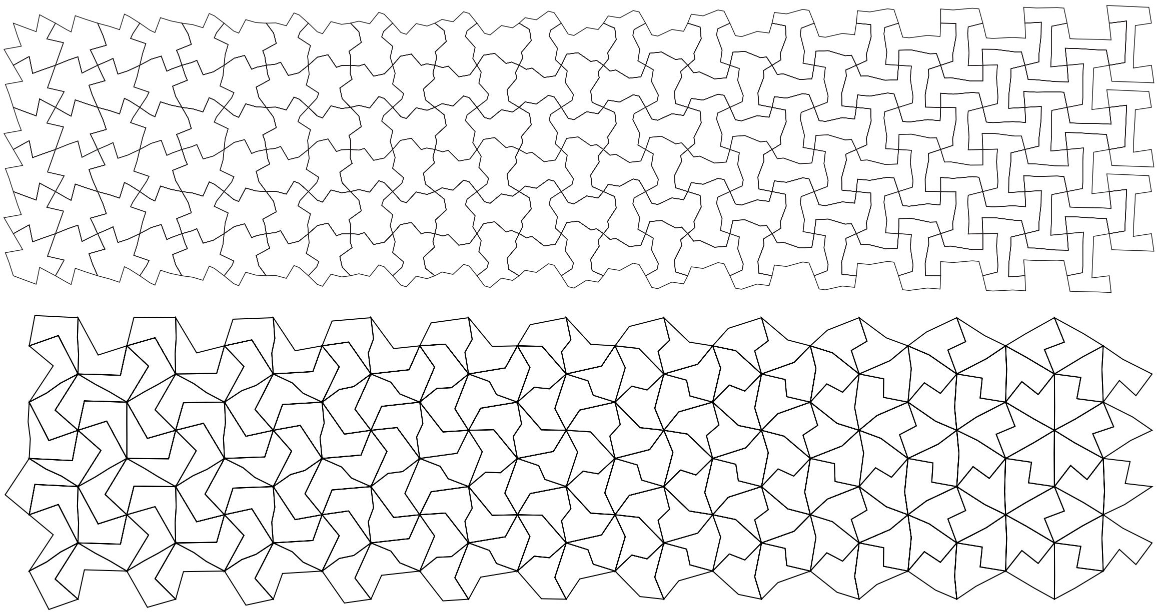


**Figure 5.2** Examples of parquet deformations.

Presumably, a one-to-one correspondence is established between the tiles of $\mathcal{T}_1$ and $\mathcal{T}_2$, and as a parameter $t$ moves from 0 to 1, each individual tile gradually deforms from its $\mathcal{T}_1$ shape to its $\mathcal{T}_2$ shape. The transition might be carried out spatially as in Escher's art, or even temporally as a smooth animation from $\mathcal{T}_1$ to $\mathcal{T}_2$.

As was mentioned in Section 3.4.1, the parquet deformations of William Huff are a kind of spatial animation. Huff was inspired directly by Escher's Metamorphoses. He distilled the style down to an abstract core, considering only interpolation transitions, and favouring abstract geometry rendered as simple line art to Escher's decorated animal forms. As reported by Hofstadter [83, Chapter 10], Huff decided further to focus on the case where $\mathcal{T}_1$ and $\mathcal{T}_2$ are "directly monohedral," in the sense that every tile is congruent to every other through translation and rotation only. We may also assume he had only periodic tilings in mind. Finally, he asked that in the intermediate stages of the deformation the tile shapes created could each be the prototile of a monohedral tiling (Hofstadter amends this rule, pointing out that some deformation might be necessary to make the intermediate shapes tile).

Inspired by parquet deformations and by Escher's interpolation transitions, we may pose the related problem of finding a smooth transition between any pair of isohedral tilings. A solution to this problem might then be expanded to encompass Escher's work (by considering a $k$-isohedral extension) or parquet deformations (by introducing the restrictions mentioned above). In any case, the isohedral problem is sufficiently interesting, and the results sufficiently attractive, that it can be fruitfully studied in isolation.

Besides choosing between temporal and spatial transitions, there is a succession of increasingly difficult problems to solve, depending on the relationship between $\mathcal{T}_1$ and $\mathcal{T}_2$:

1. The two tilings are of the same isohedral type and have congruent arrangements of tiling vertices.
2. The two tilings are of different isohedral types and have congruent arrangements of tiling vertices.
3. The two tilings are of the same isohedral type.
4. The two tilings are of the same topological type.
5. The two tilings are isohedral.

The first two cases are trivial to solve. There is a rigid motion that maps the tiling vertices of $\mathcal{T}_2$ onto the tiling vertices of $\mathcal{T}_1$, and the registration afforded by this rigid motion reduces the general interpolation of tilings to interpolation of tile edges. Any algorithm that interpolates continuously between two paths can be applied to effect a smooth transition. Linear interpolation applied to piecewise-linear tile edges can produce designs like the ones shown in Figure 5.2. Note that when the two tilings are of different isohedral types, there may be several different intermediate shapes. This situation arises when the aspects of the two tiling types do not match up, causing tiles with different relative orientations to be identified. It is not clear how to resolve this problem if we wish to satisfy Huff's goal of having a single shape at every stage in the interpolation.

The third case is easy to carry out temporally. Because the two tilings are of the same isohedral type, it is easy to create a one-to-one correspondence between their combinatoric features (tiles, tiling vertices, and tiling edges). We can then interpolate tiling vertices by linearly interpolating between the two tiling vertex parameters, and interpolate tile edges as described above. Though continuous, this interpolation may cause the tiling to undergo an arbitrary affine transform (as in the case of squares deforming into parallelograms), which does not necessarily make for a very "stable" animation.

The spatial variation of the third case is difficult. To draw the interpolation, we must first lay down an arrangement of tiling vertices that gradually changes from that of $\mathcal{T}_1$ to that of $\mathcal{T}_2$. But even within a single isohedral type, configurations of tiling vertices can change dramatically. The problem is exacerbated by the fact that the interpolation is done in the same space that the tiling is drawn. In the temporal case, there is no such interference. One possible solution is to use the underlying correspondence between tiling vertices to linearly interpolate between a tiling vertex's positions in $\mathcal{T}_1$ and $\mathcal{T}_2$. In this case, it makes sense to minimize the global affine transform between the two sets of tiling vertices, in order to make the line segment connecting any two corresponding vertices as short as possible. This minimization can be achieved using an iterated closest point algorithm such as Umeyama's [133]. Again, once the tiling vertices are laid out, tile edges can be interpolated. This approach can produce unsatisfactory results because even when the global affine transform is minimized, the interpolation can still bend and bulge, destroying the clean linear progression found in Huff's deformations. More work is needed to determine how to align the two tilings in such a way that the interpolation can be done cleanly in a strip.

The fourth case is very much like the third. Because the two tilings have the same topology, the Laves tiling with that topology can be expressed in the parameterizations of the isohedral types of both $\mathcal{T}_1$ and $\mathcal{T}_2$. This shared tiling can then be used to form the correspondence between tiling vertices, from which the previous interpolation methods follow.

The general case is the trickiest; in addition to all the difficulties encountered so far, we must account for a change in the very topology of the tiling. Thus, there can no longer be a clear correspondence between tiling vertices. On the other hand, many of Huff's examples achieve topological transitions without much effort. It seems important to handle this case to some extent.

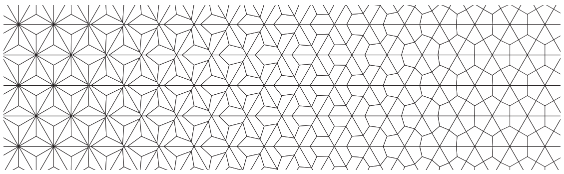
As was pointed out in Section 4.4, as a tiling edge degenerates in an isohedral tiling, the tiling undergoes a discontinuous transition to an isohedral type of a different topology. Although the transition is topologically discontinuous, it has a smooth appearance, and is therefore suitable for parquet deformations. I hypothesize that these degeneracies might be used as "gateways" to make the transition between topologies.

We can now imagine carrying out general interpolations. Given isohedral tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ of distinct topological types, we could identify a gateway tiling $\mathcal{T}_G$ that has the same topology as $\mathcal{T}_1$ and is also the degenerate case of a tiling with the same topology as $\mathcal{T}_2$. We could then build a transition by concatenating (temporally or spatially) the transitions from $\mathcal{T}_1$ to $\mathcal{T}_G$ and $\mathcal{T}_G$ (viewed now as degenerate) to $\mathcal{T}_2$.

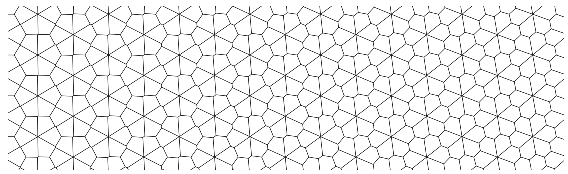
Even when it is not clear how to transition directly between two topology types, it might be possible to break the problem down into multiple steps to be assembled through concatenation. We might then reduce all topological transitions down to a set of base cases, each one a smooth transition from one Laves tiling to another. As long as any pair of Laves tilings is joined via a path of base cases, we should be able to move between any two isohedral types. I have found topological transitions that obey all the restrictions of parquet deformations and that unify all the Laves tilings except for $(4.6.12)$. These deformations are shown in Figure 5.3. I conjecture that no smooth transition is possible into or out of that tiling. Fortunately, ignoring $(4.6.12)$ leaves out exactly one isohedral type: IH77. The unreachability of that type need not be considered a significant shortcoming.

Finally, note that when multiple transitions are chained together, it has been assumed that the chaining is done through simple concatenation. This approach limits the aesthetic range of interpolation. In the temporal case, the passage through a gateway tiling may be continuous, but exhibit a jarring derivative discontinuity. In the spatial case, we would like to pass from tiling $\mathcal{T}_1$ to tiling $\mathcal{T}_2$ without having to see all the intermediate steps used to make the transition. I hypothesize that in addition to *concatenating* interpolations, we should be able to *compose* them, and have both interpolations occur simultaneously. Any sequence of interpolations could then be composed together, yielding a smooth deformation directly from one tiling to another.

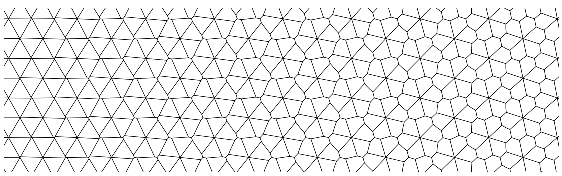
As an analogy, consider the motion of a point along a line segment. If we wish to move from position $p_1$ to $p_2$ and then from $p_2$ to $p_3$, we might simply concatenate the two trajectories; this new path will exhibit a discontinuous change in direction (and speed, if the segments have different lengths). However, de Casteljau's algorithm for drawing a quadratic Bézier curve with control points $p_1$, $p_2$, and $p_3$ short-circuits the linear trajectory and creates a smooth path that composes the two original segments. I would like to search for a method of composing tiling interpolations in the spirit of de Casteljau's algorithm.



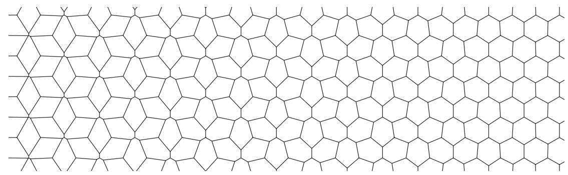
$(3.12^2)*$ $(3.4.6.4)$



$(3.4.6.4)*$ $(3^4.6)$



$(6^3)*$ $(3^4.6)$



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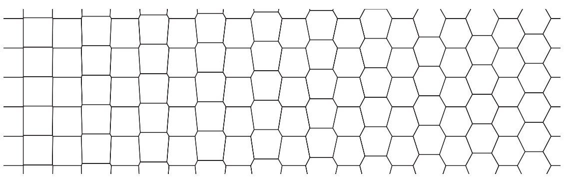
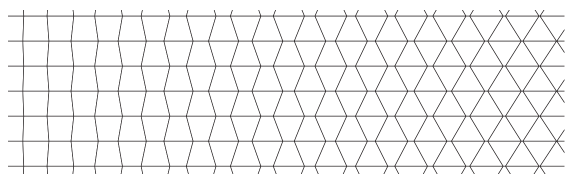
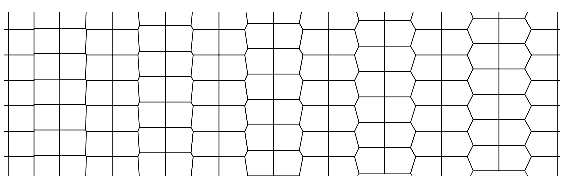


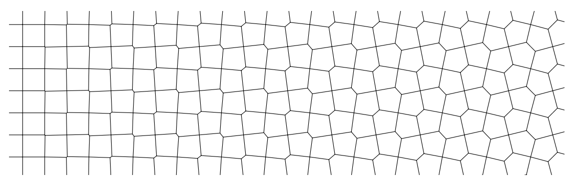
$(4^4)*$ $(3^6)$



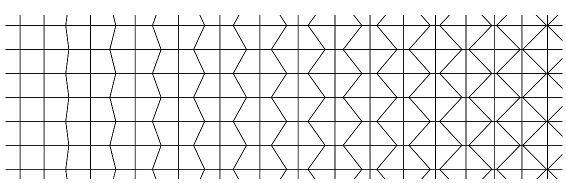
$(4^4)$ $(6^3)*$



$(4^4)*$ $(3^3.4^2)$



$(4^4)*$ $(3^2.4.3.4)$



$(4^4)$ $(4.8^2)*$

**Figure 5.3** A collection of parquet deformations between the Laves tilings. Each deformation starts and ends at a Laves tiling, as marked under the diagram. Each one will necessarily have a topological discontinuity somewhere along its length. The deformations presented here all have discontinuities at one of the endpoints; this endpoint is marked with an asterisk. By concatenating or composing these deformations, we should be able to transition between any two Laves tilings other than $(4.6.12)$.

### *5.5 A computational theory of pattern*

Symmetry is a kind of redundancy, but not all redundancy is symmetry. I have presented several examples in this dissertation of mathematical or ornamental structures for which an analysis by symmetry fails to capture some of the redundancy. Quasiperiodic tilings are a prime example; they have a tremendous amount of structure and repetition but almost no symmetry. As another example, in the construction of Islamic star patterns it was necessary to move from symmetry groups to a more localized breakdown of the plane based on tilings. Grünbaum exhibits several more examples in a paper that awakens mathematicians to the fact that symmetry is not a panacea in the study of repeated patterns [64]. In another paper [65], he criticizes the "group theory cult," a cadre of mathematicians and historians who follow Speiser in believing that the only possible characterization of order in ornament is via the group-theoretic methods of symmetry. Simply put, by expressing only the global redundancy in a pattern, symmetry fails to discern any finer structure that occurs locally.

Following more recent writing by Grünbaum on the subject [66], we may use the imprecise term "orderliness" to refer generally to structure or rules in a planar figure. Certainly, symmetry is one possible form of orderliness, but Grünbaum explores several other possible conceptions of order that do not coincide with symmetry. While it is unrealistic to seek a universal theory of orderliness, there are certainly many specific avenues that may be explored. New descriptions of order can allow us to account for more of the features of a design, and to provide a finer-grained classification of patterns than that afforded by symmetry alone.

I believe that one powerful means of understanding orderliness may lie in the study of formal languages. Symmetries are derived from a finished pattern, with no ability to see that pattern as being built step-by-step, one motif at a time. I propose to use a formal language to represent the set of legal motif placements, regarding each word in the language as actively placing a single motif rather than describing some large-scale redundancy of the pattern as a whole.

Epstein *et al.* [46, Section 2.1] provide a bridge between formal languages and group theory that can help us take the first steps away from symmetry (their development seems to owe a great deal to combinatorial group theory [108]). We start with a group $G$, a set $A$ of formal symbols, and an injective map $p$ from $A$ to $G$. We can naturally define a function $\pi$ from $A^*$, the set of all words over $A$, to $G$ by interpreting concatenation of symbols as group multiplication. If the set

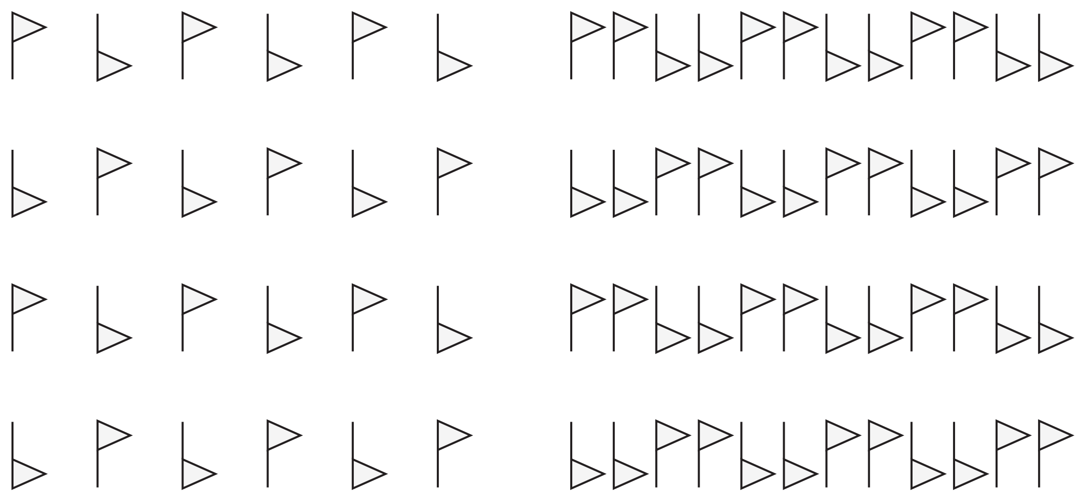
$\{p(x)|x \in A\}$ generates $G$, then $\pi$ is a surjection and we can also think of $A$ as "generating" $G$.

We can now see languages over $A$ as representing subsets of the group $G$. Of particular interest are those languages $L$ for which $\pi$ is a bijection between $L$ and $G$. The words of $L$ then correspond exactly to the elements of $G$. Working with $G$ via $L$ brings us closer to a computational view of patterns, because in practice we often "assemble" a member of $G$ by composing together a sequence of generators (*i.e.*, a word in $L$). If $L$ is a particularly well-behaved language, it might even be possible to enumerate the elements of $L$ in a useful order, which could correspond to a program that transforms a motif to every position in a pattern in a disciplined way (very much like the replication algorithms of Section 4.4.3).

The theory of automatic groups [46] shows how certain groups have languages that can be dealt with very efficiently by computer. In particular, for some groups there exists a finite automaton that can be used to enumerate the elements of the group in order by length. This automaton can then be converted into a table-driven algorithm that replicates motifs according to the $[p, q]$ symmetry groups of the hyperbolic plane [101].

The next step is to sever all ties with group theory and investigate patterns generated directly from formal languages. We define a (discrete) pattern as a motif $M$ and a language $L$ over an alphabet $A$, where each $x \in A$ corresponds to an automorphism of the plane. Every word in $L$ maps to a transformation by composing the automorphisms associated with each symbol in the word. Computational pattern theory is then the study of the properties of planar patterns that can be determined from the behaviour of these languages. We can generally consider languages $L$ that are surjective on a given pattern (several words in $L$ might map to the same transformation), or focus more precisely on bijective languages. It is also easy to work with finite pieces of a pattern through finite subsets of its language.

As an elementary example, consider the two patterns of Figure 5.4. Although clearly different, these two patterns have the same symmetry group. In (b), pairs of adjacent flags with the same orientation will be lumped together, because there is no way to recognize them as two independent, congruent motifs using symmetry. On the other hand, if $L_1$ is the language over alphabet $A_1$ corresponding to the pattern of (a), we can define $A_2 = A_1 \cup \{t\}$, where $t$ maps to a horizontal translation by half the distance between two flags in (a). It then follows that the language of the pattern in (b) can be defined over $A_2$ as $L_1 \cup L_1 t$ (where for language $L$ and symbol $x$, $Lx = \{wx|w \in L\}$).



(a) (b)

**Figure 5.4** Examples of two patterns for which symmetry groups fail to make a distinction, but formal languages might. The "egalitarian patterns" presented by Grünbaum [65] are another form of orderliness sufficiently rich to distinguish between these patterns.

Note that this outlook on the structure of (b) also seems reminiscent of Leyton's "generative theory of shape," [102] where a final shape (a pattern in our case) is expressed as a "control-nested structure." An outer group called the control group, here corresponding to $L_1$, operates on an inner fiber group, here corresponding to the set $\{\epsilon, x\}$ ($\epsilon$ is the empty word over $A_2$). In Leyton's theory, the two groups are combined using what is called a *wreath product*.

The isohedral tilings motivate a second example. In an isohedral tiling, every tile is surrounded by its neighbours in a consistent way. If the prototile has $k$ tiling edges, we can define an alphabet $H = \{h_1, \ldots, h_k\}$ where $h_i$ maps to the hop across the $i$th tiling edge (as defined in Section 4.4). Then the very simple language $H^*$ is surjective on the tiling. By contrast, a tiling that is not isohedral cannot be of the form $H^*$ for any alphabet $H$. It is slightly more complicated, but still straightforward, to give a bijective language for an isohedral tiling. Because the tiling is periodic, we can define symbols $t_1$ and $t_2$ for the two translations, and symbols $a_1, \ldots, a_m$ for the $m$ aspect transforms of the tiling (again as determined in Section 4.4). We also use $T_1$ and $T_2$ as "formal inverses" of $t_1$ and $t_2$. The formal inverses are symbols that stand for the mathematical inverses of the automorphisms associated with $t_1$ and $t_2$. A bijective regular language for the tiling is then given by the regular expression $(t_1^* \cup T_1^*)(t_2^* \cup T_2^*)(a_1 \cup a_2 \ldots \cup a_m)$. The first two factors select a particular translational unit, and the third an aspect within that unit.

The languages that can be associated with orderly patterns are more numerous than the seventeen wallpaper groups, and so it seems likely that more can be said about a pattern from its language than from its symmetry group. One especially interesting question that might benefit from a language-theoretic approach is that of measuring the information content of a pattern. The traditional tools of information theory do not apply here. However, we might consider measuring the *Kolmogorov complexity* of a pattern. Roughly speaking, the Kolmogorov complexity of a string of symbols is measured as the length of the shortest program that emits that string (when run with no input) and halts [103]. (As with classical complexity theory, we do not seek a numerical value for Kolmogorov complexity. Rather, we are interested in asymptotic results and relative complexity of different strings. Kolmogorov complexity is also useful in establishing the existence of a string with certain properties from among a collection of strings.)

It seems as if Kolmogorov complexity could be extended to patterns by considering the shortest program that loops forever, spitting out words from a language that correspond to a non-redundant enumeration of the pattern's motifs. Alternatively, it might be more fruitful to consider the shortest decision procedure for the language (a decision procedure for a language is an algorithm that is given a word and always halts, producing a yes/no answer depending on whether the word belongs to the language).

The two tilings of Figure 5.5 are a kind of converse of Figure 5.4. Here we have two tilings that are nearly the same, but for which the symmetries are vastly different. Kolmogorov complexity might help quantify the meaning of "nearly the same" here. The tiling in (b) should have only slightly more information than that of (a), because its language can be constructed by taking the language of (a), removing the words for two tiles, and grafting in rotated versions. In a similar vein, the tiling of Figure 2.10, derived in a contrived manner from the digits of $\pi$, ought to have a Kolmogorov complexity that depends fundamentally on the complexity of $\pi$ as a string.

The ideas of a computational theory of patterns and of the formal complexity of patterns raise many deep and promising questions for future work. Here are some first challenges not mentioned above:

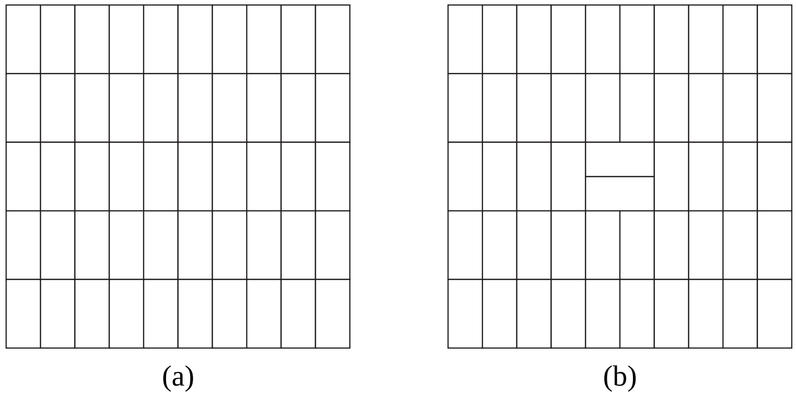



**Figure 5.5** Two tilings which would appear to have nearly the same information content, but vastly different symmetries.

- What patterns correspond to different classes of language? Every transitive pattern (*i.e.*, where the pattern's symmetries act transitively on the motifs) has a regular language. What patterns might correspond to context-free languages and to context-sensitive languages? Given a pattern, how can we determine the class of its language?

- How can we layer other traditional motif-specific information onto the language-theoretic base presented here? We should at least be able to account for multiple motif shapes and colours.

- What languages correspond to the placement of tiles in aperiodic tilings? One surjective answer for cases like the Penrose tilings is to let the symbols of an alphabet correspond to the deflation rules of the prototiles, and consider all words over the alphabet of some finite length (mapping to a uniform level of deflation everywhere). Is there a simple language for the Penrose tilings that doesn't rely on scaling?

- Can a complexity measure be used to distinguish between well-known complexity levels in patterns? In other words, if we order patterns as transitive, periodic, quasiperiodic, aperiodic, and so on, are there well defined boundaries of complexity between these classes? Can a measure of complexity help to manufacture new patterns with desired properties?