Chapter 1

INTRODUCTION

All the majesty of a city landscape
All the soaring days of our lives
All the concrete dreams in my mind's eye
All the joy I see thru' these architect's eyes.

— David Bowie

The creation of ornament is an ancient human endeavour. We have been decorating our objects, our buildings, and ourselves throughout all of history and back into prehistory. From the moment humans began to build objects of any permanence, they decorated them with patterns and textures, proclaiming beyond any doubt that the object was an artifact: a product of human workmanship. The primeval urge to decorate is bound up with the human condition.

As we evolved, so did our talents and technology for ornamentation. The history of ornament is a reflection of human history as a whole; an artifact's decoration, or lack thereof, ties it to a particular place, time, culture, and attitude.

In the last century, we have developed mathematical tools that let us peer into the past and analyze historical sources of ornament with unprecedented clarity. Even when these modern tools bear little or no resemblance to the techniques originally used to create designs, they have an undeniable explanatory power. We can then reverse the analysis process, using our newfound understanding to drive the synthesis of new designs.

Even more recently, we have crossed a threshold where these sophisticated mathematical ideas can be made eminently practical using computer technology. In the past decade, computer graphics has become ubiquitous, affordable, incredibly powerful, and relatively simple to control. The computer has become a commonplace vehicle for virtually unlimited artistic exploration, with little fear of committing unfixable errors or of wasting resources. Interactive tools give the artist instant feedback on their work; non-interactive programs can solve immense computational problems that would require considerable amounts of hand calculation or vast leaps of intuition.

The goal of this work is to seek out and exploit opportunities where modern mathematical and technological tools can be brought to bear on the analysis and synthesis of ornamental designs. The goal will be achieved by devising mathematical models for various ornamental styles, and turning those models into computer programs that can produce designs within those styles. The complete universe of ornament is obviously extremely broad, constrained only by the limits of human imagination. Therefore, I choose to concentrate here on two particular styles of ornament: Islamic star patterns and the tesselations of M. C. Escher. During these two investigations, I watch for principles and techniques that might be applied more generally to other ornamental styles.

The rest of this chapter lays the groundwork for the explorations to come, discussing the history of ornament and its analysis, and the roles played by psychology, mathematics, and computer science. In Chapter 2, I review the mathematical concepts that underlie this work. Then, the main body of research is presented: Islamic star patterns in Chapter 3, and Escher's tilings in Chapter 4. Finally, in Chapter 5, I conclude and offer ideas for future work in this area.

1.1 The study of ornament

The practice of ornament predates civilization [22]. The scholarly study and criticism of this practice is somewhat more recent, but still goes back at least to Vitruvius in ancient Rome. Gombrich provides a thorough account of the history of writings on ornament in *The Sense of Order* [60], a work that will no doubt become an important part of that history.

What is ornament? To attempt a formal definition seems ill-advised. Any precise definition will omit important classes of ornament through its narrowness, or else grow so broad as to encompass an embarrassing assortment of non-ornamental objects. In the propositions that open Jones's classic *The Grammar of Ornament* [91], we find many comments on the structure and common features of ornament, but no definition. Racinet promises to teach "more by example than by precept [121, Page 13]." Ornament, like art, is hard to pin down, always evading definition on the wings of human ingenuity.

On the other hand, the works of both Racinet and Jones teach very effectively by example. Their marvelous collections contain a multitude of designs from around the world and throughout history. Based on these collections, and the definitions that have been offered in the past, we may identify some of the more common features of ornament. We will adopt these features not as a definition, but as guidelines to make the analysis of ornament possible here.

- **Superficiality:** Jensen and Conway attribute the appeal of ornament to its "uselessness [90]." They are referring to the fact that ornament is precisely that which does not contribute to an object's function or structure. Anything that is "without use," superficial, or superfluous is an ornamental addition. As they point out, uselessness frees the designer to decorate in any way they choose, without being bound by structural or functional concerns.

- **Two-dimensionality:** Most ornament is a treatment applied to a surface. The surface may bend and twist through space, but the design upon it is fundamentally two-dimensional. A common use of ornament is as a decoration on walls, floors, and ceilings, and so adopting this restriction still leaves open many historical examples for analysis and many opportunities for synthesis.

- **Symmetry:** Symmetry is a structured form of order, balance, or repetition (it will be defined formally in Section 2.2). Speiser, one of the first mathematicians to use symmetry in studying historical ornament, required that all ornament have some degree of symmetry [135, Page 9]. This requirement seems overly strict, as there are forms of repetition that cannot be accounted for by symmetry alone, and there are many examples of ornament that repeat only in a very loose sense. Therefore I use symmetry here to refer more generally to a mathematical theory that accounts for the repetition in a particular style of ornament. At the end of this dissertation, I return to the question of the applicability of formal symmetry theory and discuss alternatives.

The history of ornamentation, particularly in the context of architecture, has been marked by the constant pull of two opposing forces. At one extreme is *horror vacui*, literally "fear of the vacuum." This term has been used to characterize the human desire to adorn every blank wall, to give every surface of a building decoration and texture. Taken to its logical conclusion, *horror vacui* produces the stereotypical Victorian parlour, saturated with ornament. A more appealing historical example is the *Book of Kells*, an illuminated Celtic manuscript whose pages are intricately ornamented (prompting Gombrich to suggest the more positive *amor infiniti* in place of *horror vacui*).

Opposing the use (and abuse) of ornament wrought by believers in *horror vacui*, we have what Gombrich calls the "cult of restraint." He uses the term to refer to those who reject ornament because of its superficiality, and praise objects that convey their essence without the need to advertise it via decoration.

The most recent revival of the cult of restraint came in the form of the modernist movement in architecture. Its pioneers were architects like Mies van der Rohe and Le Corbusier, as well as Gropius (who founded the Bauhaus in Germany) and the Italian Futurists. They rebelled against an overuse of ornament, and reveled in the beauty of technology and machines that promised to change the world for the better. To the modernists, ornament was tied to an erstwhile philosophy and way of life, and the immediate rejection of ornament was a first step to embracing the new ideals of the twentieth century [90]. Architecture of the period has a distinctly spare, austere style with blank walls and right angles.

Modernism came as a breath of fresh air after a century of stifling ornamental saturation. Unfortunately, many architects who lacked the talent of masters like Mies van der Rohe latched on to the modernist movement as a license to erect buildings in the shapes of giant, featureless concrete boxes. Thus was born yet another backlash, this time a cautious return to *horror vacui* in the form of what Jensen and Conway term *ornamentalism* [90]. Today we see some highly visible buildings that experiment with "uselessness"; a recent example is Seattle's *Experience Music Project*, designed by Frank Gehry. Overall, it seems as if the forces of modernism and ornamentalism are both active in contemporary architecture. I do not propose to sway opinion one way or the other. But if architects and other designers are willing to explore the use of geometric ornament, the work presented here could help them turn their explorations into real artifacts.

1.2 The psychology of ornament

The great majority of ornament exhibits some degree of symmetry. The reason must in part be tied to the practicalities of fabricating ornament. As a simple example, fabrics and wallpapers are printed from cylindrical templates, so their patterns will necessarily repeat in at least one direction. Looking more at the human experience of ornament, there is also a significant neurological and psychological basis for our appreciation of symmetry. This section discusses some of the reasons why there is an innate human connection between symmetry and ornament.

The science of *Psychoaesthetics* attempts to quantify our aesthetic response to sensory input. Research in psychoaesthetics shows that our aesthetic judgment of a visual stimulus derives from the arousal created and sustained by the experience of exploring and assimilating the stimulus. They test their theories by measuring physical and psychological responses of human subjects to visual stimuli.

Detection of symmetry is built in to the perceptual process at a low level. Experiments with functional brain imagining show that humans can accurately discern symmetric objects in less than one twentieth of a second [132]. The eye is particularly fast and accurate in the detection of objects with vertical mirror symmetry. The common explanation for this bias is that such symmetry might be characteristic of an advancing predator. Rapid perception can take place even across distant parts of the visual field, indicating that a large amount of mental processing is expended in locating symmetry. Furthermore, once symmetry is perceived, it is exploited. By tracking eye fixations during viewing of a scene, Locher and Nodine [106] show that in the presence of symmetry the eye will explore only non-redundant parts of that scene. Once the eye detects a line of vertical mirror symmetry, it goes on to explore only one half of the scene, the other half taken as understood.

In another experiment, Locher and Nodine show that an increase in symmetry is met with a reduction in arousal. When asked to rate appreciation of works of art, subjects rated asymmetric scenes most favourably and symmetric scenes decreasingly favourably as symmetry increased. Psychoaesthetics might help to explain this result; a more highly ordered scene requires less mental processing to assimilate, resulting in less overall engagement. While this result might appear to bode poorly for the effectiveness of symmetric ornament, mitigating factors should be considered. Most importantly, they tested the effect of symmetry by adding mirror symmetries to pre-existing works of abstract art. This wholesale modification might have destroyed other aesthetic properties of the original painting, such as its composition.

On the other hand, the reduction in arousal associated with symmetry might be appropriate for the purposes of ornamental design. In many cases, particularly in an architectural setting, the goal of ornament is to please the eye without unduly distracting it. Locher and Nodine support this claim, mentioning that as complexity of a scene increases, the rise in arousal "is pleasurable provided the increase is not enough to drive arousal into an upper range which is aversive and unpleasant [106, Page 482]."

Other research supports the correlation between symmetry and perceived goodness. In the limited domain of points in a grid, Howe [85] shows that subjective ratings of goodness correlated precisely with the degree of symmetry present. In a similar domain, Szilagyi and Baird [131] found that subjects preferred to arrange points symmetrically in a grid. In their recent review of the perception of symmetry, Møller and Swaddle simply state that humans find symmetrical objects more aesthetically pleasing than asymmetric objects [113].

Moving from the experimental side of psychology to the cognitive side, the theory of Gestalt psychology might be invoked to explain our positive aesthetic reaction to ornament. Gestalt is concerned with understanding the perceptual grouping we perform at a subconscious level when viewing a scene, and the effect this grouping has on our aesthetic response. Perhaps the most compelling explanation for the attractiveness of symmetric ornament is the "puzzle-solving" aspect of Gestalt. A symmetric pattern invites the viewer into a visual puzzle. We sense the structure on an unconscious level, and struggle to determine the rules underlying that structure. The resolution of that puzzle is a source of psychological satisfaction in the viewer. As Shubnikov and Koptsik say, "The aesthetic effects resulting from the symmetry (or other law of composition) of an object in our opinion lies in the psychic process associated with the *discovery* of its laws." [127, Page 7]

In a philosophical passage, Shubnikov and Koptsik go on to discuss the psychological and sociological effects of specific wallpaper groups [127, Page 155] (the wallpaper groups will be introduced in Section 2.2). In their theory, lines of reflection emphasize stability and rest. A line unimpeded by perpendicular reflections encourages movement. Rotational symmetries are also considered dynamic. For the various wallpaper groups, they give specific applications where ornament with those symmetries might be most appropriate.

We should not attempt to use the evidence presented in this section as a complete justification for the use of symmetry in art and ornament. But these experiments and theories reveal that we do have *some* hard-wired reaction to symmetry, a reaction that affects our perception of the world. This evidence provides us with a partial explanation for the historical importance of symmetry in ornament, and some confidence in its continued aesthetic value.

1.3 Contributions

This dissertation grew out of an open-ended exploration of the uses of computer graphics in creating geometric ornament. As such, the goals were not always stated at the outset, but were discovered along the way as my ideas developed and my techniques became more powerful. As with the artistic process in general, we cannot aim to achieve a specific goal or inspire a specific aesthetic response. But when some interesting result is found, we can then reflect on the method that produced that result and its applicability to other problems.

Here are the main contributions that this work makes to the greater world of computer graphics and computer science:

- **A model for Islamic star patterns.** The two main themes of this dissertation are presented in Chapters 3 and 4. Each of these central chapters makes a specific, thematic contribution. Chapter 3 develops a sophisticated theory that can account for the geometry of a wide range of historical Islamic star patterns. This theory is used to recreate many traditional examples, and to create novel ones.

- **A model for Escher's tilings.** Another specific contribution is the model in Chapter 4 for describing the tesselations created by M. C. Escher. The model accounts for many of the kinds of tesselations Escher created and culminates in an "Escherization" algorithm that can help an artist design novel Escher-like tesselations from scratch.

- **CAD applications.** Computer-controlled manufacturing devices are becoming ever more flexible and precise. The range of materials that can be manipulated by them is continuing to grow. Many computer scientists and engineers are investigating ways these tools can be used for scientific visualization, machining, and prototyping. I add to the list of applications by demonstrating how computer-generated ornament can be coupled with computer-aided manufacturing to produce architectural and decorative ornament quickly and easily.

- **The geometric aesthetic.** George Hart is a mathematician and sculptor who creates wonderful polyhedral sculptures in various media. He states [76] that his work "invites the viewer to partake of the geometric aesthetic." An aesthetic is a particular theory or philosophy of beauty in art. It is the set of psychological tools that allow someone to appreciate art in a particular genre or style. The geometric aesthetic is therefore a form of beauty derived primarily from the geometry of a piece of art, from its shape and the mathematical relationships among its parts. I believe that the geometric aesthetic extends beyond art to account for a feeling of elegance in mathematics. The same mindset that allows one to appreciate Hart's sculptures accounts for the sublime beauty of what Erdős called a "proof from the book," a truly ingenious and insightful proof [82].

  The work presented here is steeped in the geometric aesthetic, and in part has the goal of creating new examples of geometric art. In this regard, its contributions are intended to take part in the artistic discourse on the geometric aesthetic, to increase interest in it, and hopefully to enrich it with the many results presented here.

1.4 Other work

This section discusses some recent work by others that is generally related to the computer generation of geometric ornament. Chapters 3 and 4 each contain additional discussions of related work limited to their respective problem domains.

1.4.1 Floral ornament

An important precursor to the work in this dissertation is the paper by Wong *et al.* on floral ornament [139]. They provide a modern approach to the analysis and creation of ornament, including a taxonomy by which ornament may be classified and a "field guide" for recognizing the common features of designs. Subsequently, they develop a system capable of elaborating floral designs over finite planar regions.

Their algorithm decomposes the problem of creating floral designs into the specification of a collection of primitive motifs that make up the designs, and the elaboration of those primitives over a given region. The paper is concerned with the elaboration process, and leaves the construction of suitable motifs to the artist.

Elaboration is handled by a *growth model*, a synthetic method of distributing design elements over a region in an approximately uniform way. Growth is accomplished by applying rules to extend the design from existing motifs into currently empty parts of the region. Beginning with a set of "seeds," the algorithm iteratively applies rules until no more growth is possible. The final design can then be rendered by applying the drawing code associated with each of the motifs.

The value of the work of Wong *et al.* is that their innovations do not come at the expense of tradition. Their approach is clearly respectful of the centuries of deeply-considered thought that preceded the advent of computer graphics. Their algorithms emerge from an understanding of the intent and methods of real ornamentation, and are not developed *ex nihilo* as devices that merely appear consistent with historical examples.

For example, they eschew more traditional botanical growth models such as L-systems. The most compelling reason they give is that L-systems are a powerful tool for modeling real plants, which is exactly what floral ornament is not. There is no reason to believe that a simulation of the biological process of growth should lead to attractive designs. Their growth model represents the artist's process is creating a stylized plant design, not the growth of an actual plant.

Although the approach of Wong *et al.* lists repetition as a principle of ornament, their repetition is very loose and not constrained by global order such as symmetry. Therefore, while their results might be appropriate for an illuminated manuscript (or web page) where the surface to be decorated is small, it might be less successful in an architectural setting. Their repetition without order would deprive the viewer of any global structure to extract from the design. The visual puzzle of non-symmetric ornament is less interesting because there *is* no puzzle, only the incompressible fact of the whole design.

1.4.2 Fractals and dynamical systems

The computer has not only been used as a tool for recreating preexisting ornamental styles. Computers have also made possible styles that could not have been conceived of or executed without their capacity for precise computation and brute-force repetition.

Fractals are probably the ornamental form most closely associated with computers. They have a high degree of order, but little symmetry. The Mandelbrot set has but a single horizontal line of mirror reflection, but such a stunning degree of self-similarity that order is visible at every point and at every scale. The correspondence between parts of the Mandelbrot set is always approximate, creating an engaging visual experience. Many computer scientists continue to research interesting ways to render the Mandelbrot set and fractals like it.

Chaos is closely related to fractal geometry. Field and Golubitsky [52] have created numerous ornamental designs by plotting the attractors of dynamical systems. In particular, they have developed dynamical systems whose attractors have finite or wallpaper symmetry. In their work, we find a true rebirth of ornamental design in the digital age.

1.4.3 Celtic knotwork

The art of the Celts was always non-representational and geometric [89]. With the arrival of Christianity to their region in the middle of the first millennium C.E. came the development of the distinctive knotwork patterns most strongly associated with the Celts. Knotwork designs appear carved into tombstones, etched into personal items, and most prominently rendered in illuminated manuscripts such as the Lindisfarne Gospels and the Book of Kells. A design is formed by collections of ribbons that weave alternately over and other each other as they cross. Often, human and animal forms are intertwined with the knotwork, with ribbons becoming limbs and hair.

Celtic knotwork is the intellectual cousin of the Islamic star patterns to be discussed in Chapter 3. Both can be reduced from a richly decorated rendering to an underlying geometric description. Both are heavy users of interlacing as an aesthetic device. But most intriguing is the fact that in both cases, the historical methods of design are now lost. Research into both Celtic knotwork and Islamic star patterns has at times required the unraveling of historical mysteries.

For Celtic knotwork, one possible solution to the mystery is offered by George Bain [7], who built upon the earlier theories of J. Romilly Allen. Allen suggested that knotwork was derived from a transformation of plaitwork, the simple weave used in basketry. Bain presents a method based on breaking crossings in plaitwork and systematically rejoining the broken ribbons.

Also building on the work of Allen, Cromwell [29] presents a construction method similar to Bain's, based on an arrangement of two dual rectangular grids. Cromwell explores one-dimensional frieze patterns that appear in Celtic art and shows how the structure of generated designs relates to the arrangement of broken crossings in the underlying plaitwork.

The algorithms of Bain and Cromwell adapt readily to the computer generation of Celtic knotwork. In a series of papers, Glassner describes Bain's method and several significant extensions, creating highly attractive knotwork imagery [56, 57, 58]. Zongker [141] implemented an interactive tool similar to the one presented by Glassner. Other popular treatments of Celtic knots on the internet are given by Mercat [111] and Abbott [3].

In an interesting alternative approach, Browne [17] uses an extended tile-based algorithm to fit Celtic knots to arbitrary outlines (letterforms in his case). The technique works by filling the interior of a region with a tiling whose tiles are as close as possible to squares and equilateral triangles. Using a predefined set of tiles decorated with fragments of Celtic knotwork, he assigns motifs to tiles in such a way that the fragments link up to form a continuous Celtic knotwork design. In some cases, the result bears a strong resemblance to the illuminated letters of the ancient Celtic manuscripts. Browne's approach is certainly not the one used by the original artisans, although the final results are fairly successful.