Computer Graphics and Geometric Ornamental Design

Craig S. Kaplan

A dissertation submitted in partial fulfillment
of the requirements for the degree of

Doctor of Philosophy

University of Washington

2002

Program Authorized to Offer Degree: Computer Science & Engineering

University of Washington

Graduate School

This is to certify that I have examined this copy of a doctoral dissertation by

Craig S. Kaplan

and have found that it is complete and satisfactory in all respects,

and that any and all revisions required by the final

examining committee have been made.

Chair of Supervisory Committee:

David H. Salesin

Reading Committee:

Brian Curless

Branko Grünbaum

David H. Salesin

Date:

University of Washington

Abstract

Computer Graphics and Geometric Ornamental Design

by Craig S. Kaplan

Chair of Supervisory Committee:

Professor David H. Salesin
Computer Science & Engineering

Throughout history, geometric patterns have formed an important part of art and ornamental design. Today we have unprecedented ability to understand ornamental styles of the past, to recreate traditional designs, and to innovate with new interpretations of old styles and with new styles altogether.

The power to further the study and practice of ornament stems from three sources. We have new mathematical tools: a modern conception of geometry that enables us to describe with precision what designers of the past could only hint at. We have new algorithmic tools: computers and the abstract mathematical processing they enable allow us to perform calculations that were intractable in previous generations. Finally, we have technological tools: manufacturing devices that can turn a synthetic description provided by a computer into a real-world artifact. Taken together, these three sets of tools provide new opportunities for the application of computers to the analysis and creation of ornament.

In this dissertation, I present my research in the area of computer-generated geometric art and ornament. I focus on two projects in particular. First I develop a collection of tools and methods for producing traditional Islamic star patterns. Then I examine the tesselations of M. C. Escher, developing an "Escherization" algorithm that can derive novel Escher-like tesselations of the plane from arbitrary user-supplied shapes. Throughout, I show how modern mathematics, algorithms, and technology can be applied to the study of these ornamental styles.

TABLE OF CONTENTS

List of Figures **iii**

Chapter 1: Introduction **1**

1.1 The study of ornament . . . 2
1.2 The psychology of ornament . . . 4
1.3 Contributions . . . 7
1.4 Other work . . . 8

Chapter 2: Mathematical background **12**

2.1 Geometry . . . 12
2.2 Symmetry . . . 21
2.3 Tilings . . . 26
2.4 Transitivity of tilings . . . 37
2.5 Coloured tilings . . . 43

Chapter 3: Islamic Star Patterns **45**

3.1 Introduction . . . 45
3.2 Related work . . . 47
3.3 The anatomy of star patterns . . . 48
3.4 Hankin's method . . . 50
3.5 Design elements and the Taprats method . . . 59
3.6 Template tilings and absolute geometry . . . 67
3.7 Decorating star patterns . . . 90
3.8 Hankin tilings and Najm tilings . . . 93
3.9 CAD applications . . . 98

3.10 Nonperiodic star patterns 103
3.11 Future work 111

Chapter 4: Escher's Tilings 116

4.1 Introduction 116
4.2 Related work 118
4.3 Parameterizing the isohedral tilings 119
4.4 Data structures and algorithms for IH 125
4.5 Escherization 135
4.6 Dihedral Escherization 149
4.7 Non-Euclidean Escherization 166
4.8 Discussion and future work 175

Chapter 5: Conclusions and Future work 182

5.1 Conventionalization 182
5.2 Dirty symmetry 184
5.3 *Snakes* 185
5.4 Deformations and metamorphoses 187
5.5 A computational theory of pattern 195

Bibliography 200

LIST OF FIGURES

2.1 Trigonometric identities for a right triangle in absolute geometry 20

2.2 Figures with finite discrete symmetry groups 24

2.3 Examples of symmetry groups of the form $[p, q]$ 25

2.4 A tiling for which some tiles intersect in multiple disjoint curves 27

2.5 The features of a tiling with polygonal tiles 28

2.6 The eleven Archimedean tilings 30

2.7 Some Euclidean and non-Euclidean regular tilings 31

2.8 The eleven Laves tilings 33

2.9 The two famous aperiodic Penrose tilings 34

2.10 A simple aperiodic tiling 35

2.11 Sample matching conditions on the rhombs of Penrose's aperiodic tile set $P3$ . . . 36

2.12 An example of a monohedral tiling that is not isohedral 38

2.13 Heesch's anisohedral prototile 39

2.14 The behaviour of different isohedral tiling types under a change to one tiling edge . 40

2.15 An example of an isohedral tiling of type IH16 41

2.16 Five steps in the derivation of an isohedral tiling's incidence symbol 42

3.1 The rays associated with a contact position in Hankin's method 52

3.2 A demonstration of Hankin's method 53

3.3 Examples of star patterns constructed using Hankin's method 54

3.4 Two extensions to the basic inference algorithm 55

3.5 Using the δ parameter to enrich Hankin's method 56

3.6 Examples of two-point patterns constructed using Hankin's method 56

3.7 The construction of an Islamic parquet deformation based on Hankin's method . . 58

3.8 More examples of Islamic parquet deformation based on Hankin's method 58
3.9 The discovery of a complex symmetric motif in a star pattern 59
3.10 Path-based construction of radially-symmetry design elements 61
3.11 Examples of stars 63
3.12 Examples of rosettes 63
3.13 A diagram used to explain the construction of Lee's ideal rosette 65
3.14 Two diagrams used to explain the construction of generalized rosettes 65
3.15 Examples of rosettes 66
3.16 The extension process for design elements 66
3.17 A visualization of how Taprats assembles a star pattern 68
3.18 Examples of designs constructed using Taprats 69
3.19 The canonical triangle used in the construction of Najm tilings 72
3.20 Examples of e and v orientations for regular polygons 73
3.21 An example of constructing a template tiling in absolute geometry 74
3.22 Examples of tilings that can be constructed using the procedure and notation given in Section 3.6.1 79
3.23 Examples of symmetrohedra 80
3.24 A diagram used to build extended motifs in absolute geometry 81
3.25 An excerpt from the absolute geometry library underlying Najm, showing the class specialization technique 84
3.26 Samples of Islamic star patterns that can be produced using Najm 85
3.27 Examples of decoration styles for star patterns 91
3.28 A diagram used to compute the mitered join of two line segments in absolute geometry 92
3.29 Examples of distinct tilings that can produce the same Islamic design 94
3.30 The rosette transform applied to a regular polygon 95
3.31 The rosette transform applied to an irregular polygon 95
3.32 Two demonstrations of how a simpler Taprats tiling is turned into a more complex Hankin tiling 96

3.33 An example of how a Najm tiling, combined with richer design elements, can produce superior designs to Hankin's method alone . . . 98
3.34 An unusual star pattern with 11- and 13-pointed stars . . . 99
3.35 Examples of laser-cut star patterns . . . 100
3.36 Examples of star patterns created using a CNC milling machine . . . 102
3.37 Examples of waterjet-cut star patterns . . . 102
3.38 Examples of star patterns fabricated using rapid prototyping tools . . . 103
3.39 Examples of monster polygons . . . 105
3.40 A menagerie of monsters and their motifs . . . 106
3.41 The development of design fragments for a Quasitiler-based Islamic star pattern . . . 107
3.42 Examples of Quasitiler-based Islamic star patterns . . . 108
3.43 The construction of Kepler's Aa tiling . . . 110
3.44 Proposed modifications to the region surrouding the pentacle in Kepler's Aa tiling that permit better inference of motifs . . . 111
3.45 Examples of star patterns based on Kepler's Aa tiling . . . 112

4.1 M.C. Escher in a self-portrait . . . 116
4.2 Examples of **J**, **U**, **S** and **I** edges . . . 120
4.3 The complete set of tiling vertex parameterizations for the isohedral tilings . . . 122
4.4 The derivation of a tiling vertex parameterization for one isohedral type . . . 124
4.5 The effect of varying the tiling vertex parameters . . . 125
4.6 Template information for one isohedral type . . . 126
4.7 A visualization of how isohedral tilings are coloured . . . 127
4.8 A visualization of the `rules` section of an isohedral template . . . 128
4.9 Sample code implementing a tiling vertex parameterization . . . 130
4.10 An example of how a degenerate tile edge leads to a related tiling of a different isohedral type . . . 131
4.11 A screen shot from Tactile, the interactive viewer and editor for isohedral tilings . . . 134
4.12 The replication algorithm for periodic Euclidean tilings . . . 134

4.13 Timelines for two sample Escherization runs 142
4.14 The result of user editing of an Escherized tile 142
4.15 Examples of Escherization 143
4.16 An example of a 2-isohedral tiling with different numbers of A and B tiles 151
4.17 A summary of split isohedral Escherization 152
4.18 Examples of split isohedral escherization 154
4.19 Examples of Heaven and Hell Escherization 156
4.20 An example of a Sky and Water design, based on the goal shapes of Figure 4.18(d) 159
4.21 An example of how tiling vertices can emerge in Penrose's aperiodic set P2 162
4.22 A tiling vertex parameterization for Penrose's aperiodic set $P2$ 163
4.23 A tiling vertex parameterization for Penrose's aperiodic set $P3$ 164
4.24 Edge labels for the tiling edges of the two sets of Penrose tiles, in the spirit of the incidence symbols used for the isohedral tilings 165
4.25 Examples of dihedral Escherization using Penrose's aperiodic set P2 167
4.26 Examples of tesselations based on Penrose's aperiodic set P3 168
4.27 A visualization of a tiling's symmetry group 172
4.28 The mapping from a square texture to the surface of a sphere 173
4.29 The mapping from a square texture to a quadrilateral in the hyperbolic plane 173
4.30 An uncoloured interpretation of "Tea-sselation" mapped into the hyperbolic plane with symmetry group $[4,5]^+$ 174
4.31 A coloured interpretation of "Tea-sselation" mapped into the hyperbolic plane with symmetry group $[4,5]^+$ 176
4.32 An Escherized tiling mapped onto the sphere 177
4.33 A coloured interpretation of "Tea-sselation" mapped into the hyperbolic plane with symmetry group $[4,6]^+$ 177
4.34 A simulated successor to Escher's *Circle Limit* drawings 178
4.35 A goal shape for which Escherization performs badly 178

5.1 A visualization of the geometric basis of Escher's *Snakes* 186

5.2 Examples of parquet deformations 190
5.3 A collection of parquet deformations between the Laves tilings 194
5.4 Examples of two patterns for which symmetry groups fail to make a distinction, but formal languages might 197
5.5 Two tilings which would appear to have nearly the same information content, but vastly different symmetries 199

ACKNOWLEDGMENTS

The work in this dissertation extends and elaborates on the earlier results of three papers. The research in Chapter 3 on Islamic star patterns grew out of work first presented at the 2000 Bridges conference [93], and later reprinted in the online journal *Visual Mathematics* [94]. The template tilings of Section 3.6.1 were first described in a paper co-authored with George Hart and presented at the 2001 Bridges conference [95]. The work on Escher's tilings and Escherization made its debut in a paper co-authored with David Salesin and presented at the 2000 SIGGRAPH conference [96]. A great deal of work that followed from these papers appears here for the first time.

This work has integrated ideas from diverse fields within and outside of computer science. So many people have contributed their thoughts and insights over the years that it seems certain I will overlook someone below. For any omissions, I can only apologize in advance.

My work on Islamic star patterns began as a final project in a course on Islamic art taught by Mamoun Sakkal in the spring of 1999. Mamoun provided crucial early guidance and motivation in my pursuit of star patterns. Jean-Marc Castéra has also been a source of insights and ideas in this area. More recently, I have benefited greatly from contact with Jay Bonner; my understanding of star patterns took a quantum leap forward after working with him.

I had always wanted to explore the CAD applications of star patterns. It was because of the enthusiasm and generosity of Nathan Myhrvold that I finally had the opportunity.

Collaboration with Nathan led to a variety of other CAD experiments. These experiments usually involved the time, grace, and physical resources of others, and so I must thank those who helped with building real-world artifacts: Carlo Séquin (rapid prototyping), Keith Ritala and Eric Miller (laser cutting), Seth Green (CNC milling), and James McMurray (Solidscape prototyping and, hopefully, metal casting).

My work on tilings began as a final project in a computer graphics course, and might therefore never have gotten off the ground without the hard work of my collaborators on that project, Michael

Noth and Jeremy Buhler. Escherization relied on crucial ideas from Branko Grünbaum, Michael Ernst, and John Hughes, and the valuable input of Tony DeRose, Zoran Popović, Dan Huttenlocher, and Olaf Delgado-Friedrichs. Doug Dunham helped me with the basics of non-Euclidean geometry, and therefore had a profound effect on my research into both Escher tilings and Islamic star patterns.

I am fortunate to have authored papers with a small but extremely talented group of people. George Hart, Erik Demaine, and Martin Demaine established a pace and quality of research that I can only hope to have internalized through our interactions. The same holds true for Craig Chambers and Michael Ernst, with whom I published earlier work in the field of programming languages.

Certain individuals stand out as having gone out of their way to provide advice and encouragement, acting as champions of my work. I owe special thanks in this regard to John Hughes and Victor Ostromoukhov. Thanks also to the amazingly energetic Reza Sarhangi.

The members of my supervisory committee (Brian Curless, Andrew Glassner, Branko Grünbaum, Zoran Popović and David Salesin) were invaluable. They provided a steady stream of insights, suggestions, advice, and brainstorms. My reading committee, made up of David, Brian, and Branko, brought about significant improvements to this document through their careful scrutiny.

The work presented here would certainly not have been possible without the influence of my advisor and friend, David Salesin. I could never have guessed that because of him, I would make a career of something that felt so much like play. From David, I have learned a great deal about choosing problems, about doing research, and about communicating the results.

The countless hours I spent at the computer in pursuit of this research would have been quite painful without the many open source tools and libraries that I take for granted every day. Thank you to the authors of Linux and the numerous software packages that run on it. Many Escherization experiments relied on photographs for input; most of these were drawn from the archive of free images at `freefoto.com`.

Many thanks to Margareth Verbakel and Cordon Art B.V. for their generosity and indulgence in allowing me to reprint Escher's art in this dissertation. More information on reprinting Escher's art can be found at `www.mcescher.com`.

I have never once regretted my decision to come to the University of Washington. It has been an honour to work in the department’s friendly, supportive, and cooperative environment. Thanks in general to all my friends among the faculty, staff, and students, and in particular to Doug Zongker for more lunch conversation than I could possibly count.

Thank you to my parents for their unflagging dedication and absolute confidence in me, and to my brother, grandmother, and extended family for their perpetual care and support.

Finally, it is impossible to believe that I might have completed this work without the support, advice, generosity, devotion, and occasional skepticism of my wife Nathalie. To her, and to my daughter Zoë, I owe my acknowledgment, my dedication, and my love.

BIBLIOGRAPHY

[1] Jan Abas. A hyperbolic mural. http://www.bangor.ac.uk/~mas009/ppic7.html.

[2] Syed Jan Abas and Amer Shaker Salman. *Symmetries of Islamic Geometrical Patterns*. World Scientific, 1995.

[3] Steve Abbott. Steve Abbott's computer drawn Celtic knotwork. http://www.abbott.demon.co.uk/knots.html.

[4] Maneesh Agrawala and Chris Stolte. Rendering effective route maps: Improving usability through generalization. *Proceedings of SIGGRAPH 2001*, pages 241–250, 2001.

[5] Nina Amenta and Mark Phillips. Kali. http://www.geom.umn.edu/java/Kali/.

[6] E. M. Arkin, L. P. Chew, D. P. Huttenlocher, K. Kedem, and J. S. B. Mitchell. An efficiently computable metric for comparing polygonal shapes. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 13:209–216, 1991.

[7] George Bain. *The Methods of Construction of Celtic Art*. Dover, 1973.

[8] Thaddeus Beier and Shawn Neely. Feature-based image metamorphosis. *Proceedings of SIGGRAPH'92*, pages 35–42, 1992.

[9] John Berglund. Anisohedral tilings page. http://www.angelfire.com/mn3/anisohedral/.

[10] John Berglund. Is there a k-anisohedral tile for $k \geq 5$? *American Mathematical Monthly*, 100:585–588, 1993.

[11] Jinny Beyer. *Designing Tesselations: The Secrets of Interlocking Patterns*. Contemporary Books, 1999.

[12] Jay Bonner. Geodazzlers. http://www.dstoys.com/GD.html.

[13] Jay Francis Bonner. *Islamic Geometric Patterns: Their Historical Development and Traditional Methods of Derivation*. Unpublished, 2000.

[14] Roberto Bonola. *Non-Euclidean Geometry*. Dover Publications, 1955.

[15] F. H. Bool, J. R. Kist, J. L. Locher, and F. Wierda. *M. C. Escher: His Life and Complete Graphic Work*. Harry N. Abrams, Inc., 1992.

[16] J. Bourgoin. *Arabic Geometrical Pattern and Design*. Dover Publications, 1973.

[17] Cameron Browne. Font decoration by automatic mesh fitting. In R.D. Hersch, J. Andr, and H. Brown, editors, *Electronic Publishing, Artistic Imaging, and Digital Typography*, pages 23–43. Springer Verlag, 1998.

[18] Paul Burchard, Daeron Meyer, and Eugenio Durand. Quasitiler. `http://www.geom.umn.edu/apps/quasitiler/`.

[19] Jean-Marc Castéra. *Arabesques: Decorative Art in Morocco*. ACR Edition, 1999.

[20] Jean-Marc Castéra. Zellijs, muqarnas and quasicrystals. In Nathaniel Friedman and Javiar Barrallo, editors, *ISAMA 99 Proceedings*, pages 99–104, 1999.

[21] William W. Chow. Automatic generation of interlocking shapes. *Computer Graphics and Image Processing*, 9:333–353, 1979.

[22] Archibald Christie. *Traditional Methods of Pattern Designing*. Oxford University Press, 1929.

[23] Alain Connes. *Noncommutative Geometry*. Academic Press, San Diego, 1994.

[24] H. S. M. Coxeter. Coloured symmetry. In H. S. M. Coxeter *et al.*, editor, *M.C. Escher: Art and Science*, pages 15–33. Elsevier Science Publishers B.V., 1986.

[25] H. S. M. Coxeter. *Non-Euclidean Geometry*. Mathematical Association of America, sixth edition, 1998.

[26] H. S. M. Coxeter and W. O. J. Moser. *Generators and Relations for Discrete Groups*. Springer-Verlag, 1980.

[27] Keith Critchlow. *Islamic Patterns: An Analytical and Cosmological Approach*. Thames and Hudson, 1976.

[28] Hallard T. Croft, Kenneth J. Falconer, and Richard K. Guy. *Unsolved Problems in Geometry*. Springer-Verlag, 1991.

[29] Peter R. Cromwell. Celtic knotwork: Mathematical art. *The Mathematical Intelligencer*, 15(1):36–47, 1993.

[30] L. Danzer, B. Grünbaum, and G. C. Shephard. Can all tiles of a tiling have five-fold symmetry? *American Mathematical Monthly*, 89:568–585, 1982.

[31] Prisse d'Avennes. *Arabic Art*. L'Aventurine, 2001.

[32] Doug DeCarlo and Anthony Santella. Stylization and abstraction of photographs. *Proceedings of SIGGRAPH 2002*, pages 769–776, 2002.

[33] Olaf Delgado, Daniel Huson, and Elizaveta Zamorzaeva. The classification of 2-isohedral tilings of the plane. *Geometricae Dedicata*, 42:43–117, 1992.

[34] Olaf Delgado-Friedrichs. Data structures and algorithms for tilings. To be published in *Theoretical Computer Science*.

[35] Tony DeRose, Michael Kass, and Tien Truong. Subdivision surfaces in character animation. In *Proceedings of the 25th annual conference on Computer graphics and interactive techniques (SIGGRAPH'98)*, pages 85–94. ACM Press, 1998.

[36] A.K. Dewdney. *The Tinkertoy Computer and Other Machinations*, pages 222–230. W. H. Freeman, 1993.

[37] Robert Dixon. Two conformal mappings. In Michele Emmer, editor, *The Visual Mind: Art and Mathematics*, pages 45–48. MIT Press, 1993.

[38] Andreas W. M. Dress. The 37 combinatorial types of regular "Heaven and Hell" patterns in the euclidean plane. In H. S. M. Coxeter *et al.*, editor, *M.C. Escher: Art and Science*, pages 35–45. Elsevier Science Publishers B.V., 1986.

[39] Douglas Dunham. Hyperbolic symmetry. *Computers and Mathematics with Applications*, 12B(1/2):139–153, 1986.

[40] Douglas Dunham. Artistic patterns in hyperbolic geometry. In Reza Sarhangi, editor, *Bridges 1999 Proceedings*, pages 139–149, 1999.

[41] Douglas Dunham. Hyperbolic Islamic patterns — a beginning. In Reza Sarhangi, editor, *Bridges 2001 Proceedings*, pages 247–254, 2001.

[42] Douglas Dunham, John Lindgren, and David Witte. Creating repeating hyperbolic patterns. *Computer Graphics (Proc. SIGGRAPH)*, pages 215–223, 1981.

[43] Douglas J. Dunham. Creating hyperbolic escher patterns. In H. S. M. Coxeter *et al.*, editor, *M.C. Escher: Art and Science*, pages 241–247. Elsevier Science Publishers B.V., 1986.

[44] David S. Ebert, F. Kenton Musgrave, Darwyn Peachey, Ken Perlin, and Steven Worley. *Texturing and Modeling*. AP Professional, 1998.

[45] David Eppstein. The geometry junkyard: Penrose tiling. `http://www.ics.uci.edu/~eppstein/junkyard/penrose.html`.

[46] David B. A. Epstein, J .W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston. *Word Processing in Groups*. Jones and Bartlett, 1992.

[47] Bruno Ernst. *The Magic Mirror of M. C. Escher*. Ballantine Books, New York, 1976.

[48] George A. Escher. M.C. escher at work. In H. S. M. Coxeter *et al.*, editor, *M.C. Escher: Art and Science*, pages 1–8. Elsevier Science Publishers B.V., 1986.

[49] M. C. Escher. *Escher on Escher: Exploring the Infinite*. Henry N. Abrams, Inc., 1989. Translated by Karin Ford.

[50] Richard L. Faber. *Foundations of Eucliedean And Non-Euclidean Geometry*. Marcel Dekker, Inc., 1993.

[51] Michael Field and Martin Golubitsky. *Symmetry in Chaos*. Oxford University Press, 1992.

[52] François Dispot. Arabeske home page. `http://www.wozzeck.net/arabeske/index.html`.

[53] The gimp. `http://www.gimp.org/`.

[54] Andrew Glassner. Andrew Glassner's notebook: Aperiodic tiling. *IEEE Computer Graphics & Applications*, 18(3):83–90, may–jun 1998. ISSN 0272-1716.

[55] Andrew Glassner. Andrew Glassner's notebook: Penrose tiling. *IEEE Computer Graphics & Applications*, 18(4), jul–aug 1998. ISSN 0272-1716.

[56] Andrew Glassner. Andrew Glassner's notebook: Celtic knots, part II. *IEEE Computer Graphics & Applications*, 19(6):82–86, nov–dec 1999. ISSN 0272-1716.

[57] Andrew Glassner. Andrew Glassner's notebook: Celtic knotwork, part I. *IEEE Computer Graphics & Applications*, 19(5):78–84, sep–oct 1999. ISSN 0272-1716.

[58] Andrew Glassner. Andrew Glassner's notebook: Celtic knots, part III. *IEEE Computer Graphics & Applications*, 20(1):70–75, jan–feb 2000. ISSN 0272-1716.

[59] Solomon W. Golomb. *Polyominoes: Puzzles, Patterns, Problems and Packings*. Princeton University Press, second edition, 1994.

[60] E. H. Gombrich. *The Sense of Order: A Study in the Psychology of Decorative Art*. Phaidon Press Limited, second edition, 1998.

[61] Chaim Goodman-Strauss. A small set of aperiodic tiles. *European Journal of Combinatorics*, 20:375–384, 1999.

[62] Marvin J. Greenberg. *Euclidean and Non-Euclidean Geometries: Development and History*. W. H. Freeman and Company, third edition, 1993.

[63] Bathsheba Grossman. Bathsheba Grossman – sculpting geometry. `http://www.bathsheba.com/`.

[64] Branko Grünbaum. The emperor's new clothes: Full regalia, G string, or nothing? *The Mathematical Intelligencer*, 6(4):47–53, 1984.

[65] Branko Grünbaum. Periodic ornamentation of the fabric plane: Lessons from Peruvian fabrics. *Symmetry*, 1(1):45–68, 1990.

[66] Branko Grünbaum. Levels of orderliness: Global and local symmetry. In I. Hargittai and T. C. Laurent, editors, *Symmetry 2000*, pages 51–61. Portland Press, London, 2002.

[67] Branko Grünbaum and G. C. Shephard. Spherical tilings with transitivity properties. In Chandler Davis, Branko Grünbaum, and F. A. Sherk, editors, *The Geometric Vein: The Coxeter Festschrift*, pages 65–94. Springer-Verlag, New York, 1982.

[68] Branko Grünbaum and G. C. Shephard. *Tilings and Patterns*. W. H. Freeman, 1987.

[69] Branko Grünbaum and G. C. Shephard. Interlace patterns in Islamic and Moorish art. *Leonardo*, 25:331–339, 1992.

[70] E. H. Hankin. *The Drawing of Geometric Patterns in Saracenic Art*, volume 15 of *Memoirs of the Archaeological Society of India*. Government of India, 1925.

[71] E. Hanbury Hankin. Examples of methods of drawing geometrical arabesque patterns. *The Mathematical Gazette*, pages 371–373, May 1925.

[72] E. Hanbury Hankin. Some difficult Saracenic designs II. *The Mathematical Gazette*, pages 165–168, July 1934.

[73] E. Hanbury Hankin. Some difficult Saracenic designs III. *The Mathematical Gazette*, pages 318–319, December 1936.

[74] István Hargittai and Magdolna Hargittai. *Symmetry through the Eyes of a Chemist*. Plenum Press, New York, second edition, 1995.

[75] George W. Hart. Conway notation for polyhedra. http://www.georgehart.com/virtual-polyhedra/conway_notation.html.

[76] George W. Hart. Geometric sculpture. http://www.georgehart.com/sculpture/sculpture.html.

[77] George W. Hart. Uniform polyhedra. http://www.georgehart.com/virtual-polyhedra/uniform-info.html.

[78] Barbara Hausmann, Britta Slopianka, and Hans-Peter Seidel. Exploring plane hyperbolic geometry. In Hans-Christian Hege and Konrad Polthier, editors, *Visualization and Mathematics*, pages 21–36. Springer, 1997.

[79] H. Heesch. Aufbau der ebene aus kongruenten bereichen. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen*, pages 115–117, 1935. John Berglund provides an online English translation at http://www.angelfire.com/mn3/anisohedral/heesch35.html.

[80] H. Heesch and O. Kienzle. *Flachenschluss*. Springer–Verlag, 1963.

[81] Richard E. Hodel. *An Introduction to Mathematical Logic*. PWS Publishing Company, 1995.

[82] Paul Hoffman. *The Man Who Loved Only Numbers*. Hyperion, New York, 1998.

[83] Douglas Hofstadter. *Metamagical Themas: Questing for the Essence of Mind and Pattern*. Bantam Books, 1986.

[84] Alan Holden. *Shapes, Space, and Symmetry*. Dover Publications, Inc., 1991.

[85] Edmund S. Howe. Effects of partial symmetry, exposure time, and backward masking on judged goodness and reproduction of visual patterns. *Quarterly Journal of Experimental Psychology*, 32:27–55, 1980.

[86] Daniel H. Huson. The generation and classification of tile-k-transitive tilings of the euclidean plane, the sphere, and the hyperbolic plane. *Geometriae Dedicata*, 47:269–296, 1993.

[87] Pedagoguery Software Inc. Tess. http://www.peda.com/tess/Welcome.html.

[88] Washington Irving. *The Alhambra*. Macmillan and co., 1931.

[89] Drew Ivans. The origin and meaning of Celtic knotwork. http://www.craytech.com/drew/knotwork/knotwork-meaning.html.

[90] Robert Jensen and Patricia Conway. *Ornamentalism: the New Decorativeness in Artchitecture & Design*. Clarkson N. Potter, Inc., 1982.

[91] Owen Jones. *The Grammar of Ornament*. Studio Editions, 1986.

[92] Craig S. Kaplan. `isohedral.ih`. `http://www.cs.washington.edu/homes/csk/tile/escherization.html`.

[93] Craig S. Kaplan. Computer generated islamic star patterns. In Reza Sarhangi, editor, *Bridges 2000 Proceedings*, 2000.

[94] Craig S. Kaplan. Computer generated islamic star patterns. *Visual Mathematics*, 2(3), 2000. `http://members.tripod.com/vismath4/kaplan/index.html`.

[95] Craig S. Kaplan and George W. Hart. Symmetrohedra: Polyhedra from symmetric placement of regular polygons. In Reza Sarhangi, editor, *Bridges 2001 Proceedings*, 2001.

[96] Craig S. Kaplan and David H. Salesin. Escherization. In *Proceedings of the 27th annual conference on Computer graphics and interactive techniques (SIGGRAPH 2000)*, pages 499–510. ACM Press/Addison-Wesley Publishing Co., 2000.

[97] David C. Kay. *College Geometry*. Holt, Rinehart and Winston, Inc., 1969.

[98] Haresh Lalvani. Pattern regeneration: A focus on Islamic jalis and mosaics. In Jan Pieper and George Michell, editors, *The Impulse to Adorn*, pages 123–136. Marg Publications, 1982.

[99] A.J. Lee. Islamic star patterns. *Muqarnas*, 4:182–197, 1995.

[100] Raph Levien. libart. `http://www.levien.com/libart/`.

[101] Silvio Levy. Automatic generation of hyperbolic tilings. In Michele Emmer, editor, *The Visual Mind: Art and Mathematics*, pages 165–170. MIT Press, 1993.

[102] Michael Leyton. *A Generative Theory of Shape*, volume 2145 of *Lecture Notes in Computer Science*. Springer, 2001.

[103] Ming Li and Paul Vitányi. *An Introduction to Kolmogorov Complexity and its Applications*. Graduate texts in Computer Science. Springer-Verlag, Inc., New York, second edition, 1997.

[104] Stanley B. Lippman and Josée Lajoie. *C++ Primer*. Addison-Wesley, third edition, 1998.

[105] J. L. Locher. *The Magic of M. C. Escher*. Harry N. Abrams, Inc., 2000. Designed by Erik Thé.

[106] P. Locher and C. Nodine. The perceptual value of symmetry. *Computers and Mathematics With Applications*, 17(4–6):475–484, 1989.

[107] Z. Lučić and E. Molnár. Fundamental domains for planar discontinuous groups and uniform tilings. *Geometriae Dedicata*, 40:125–143, 1991.

[108] Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial group theory: presentations of groups in terms of generators and relations*. Dover Publications, 1976.

[109] George E. Martin. *The Foundations of Geometry and the Non-Euclidean Plane*. Intext Educational Publishers, 1975.

[110] Scott McCloud. *Understanding Comics*. HarperCollins, 1993.

[111] Christian Mercat. Celtic knotwork, the ultimate tutorial. `http://www.entrelacs.net/en.index.html`.

[112] Kevin Mitchell. Semi-regular tilings of the plane. `http://people.hws.edu/mitchell/tilings/Part1.html`.

[113] Anders Pape Møller and John P. Swaddle. The biological importance of imperfect symmetry. In *Asymmetry, Developmental Stability and Evolution*, chapter W1. Oxford University Press, 1997. Available online from `http://www.oup.co.uk/isbn/0-19-854894-X`.

[114] E. N. Mortenson and W. A. Barrett. Intelligent scissors for image composition. *Proceedings of SIGGRAPH 1995*, pages 191–198, 1995.

[115] Joseph Myers. Polyomino, polyhex, and polyiamond tiling. `http://www.srcf.ucam.org/~jsm28/tiling/`.

[116] Gülru Necipoğlu. *The Topkapı Scroll — Geometry and Ornament in Islamic Architecture*. The Getty Center for the History of Art and the Humanities, 1995.

[117] John Allen Paulos. *Mathematics and Humor*. The University of Chicago Press, 1980.

[118] Roger Penrose. Escher and the visual representation of mathematical ideas. In H. S. M. Coxeter *et al.*, editor, *M.C. Escher: Art and Science*, pages 143–157. Elsevier Science Publishers B.V., 1986.

[119] Ivars Peterson. *The Mathematical Tourist: Snapshots of Modern Mathematics*. W. H. Freeman and Company, New York, 1988.

[120] William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery. *Numerical Recipes in C: The Art of Scientific Computing*. Cambridge University Press, second edition, 1992. ISBN 0-521-43108-5. Held in Cambridge.

[121] A. Racinet. *The Encyclopedia of Ornament*. Studio Editions, 1988.

[122] J. F. Rigby. Butterflies and snakes. In H. S. M. Coxeter *et al.*, editor, *M.C. Escher: Art and Science*, pages 211–220. Elsevier Science Publishers B.V., 1986.

[123] Doris Schattschneider. Escher's classification system for his colored periodic drawings. In H. S. M. Coxeter *et al.*, editor, *M.C. Escher: Art and Science*, pages 82–96. Elsevier Science Publishers B.V., 1986.

[124] Doris Schattschneider. *M.C. Escher: Visions of Symmetry*. W.H. Freeman, 1990.

[125] P. J. Schneider. An algorithm for automatically fitting digitized curves. In *Graphics Gems*, pages 612–626. Academic Press, Boston, 1990.

[126] G. C. Shephard. What Escher might have done. In H. S. M. Coxeter *et al.*, editor, *M.C. Escher: Art and Science*, pages 111–122. Elsevier Science Publishers B.V., 1986.

[127] A. V. Shubnikov and V. A. Koptsik. *Symmetry in Science and Art*. Plenum Press, 1974.

[128] Peter Stampfli. New quasiperiodic lattices from the grid method. In István Hargittai, editor, *Quasicrystals, Networks and Molecules of Fivefold Symmetry*, chapter 12, pages 201–221. VCH Publishers, 1990.

[129] Desmond Stewart. *The Alhambra*. Newsweek, 1974.

[130] Ian Stewart. *The Problems of Mathematics*. Oxford University Press, 1987.

[131] Peter G. Szilagyi and John C. Baird. A quantitative approach to the study of visual symmetry. *Perception & Psychophysics*, 22(3):287–292, 1977.

[132] C. W. Tyler. The human expression of symmetry: Art and neuroscience. `http://www.ski.org/CWTyler_lab/CWTyler/Art\%20Investigations/Symmetry/S%ymmetry.html`.

[133] Shinji Umeyama. Least-squares estimation of transformation parameters between two point patterns. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 13(4):376–380, April 1991.

[134] David Wade. *Pattern in Islamic Art*. The Overlook Press, 1976.

[135] Dorothy K. Washburn and Donald W. Crowe. *Symmetries of Culture*. University of Washington Press, 1992.

[136] Magnus J. Wenninger. *Polyhedron Models*. Cambridge University Press, 1971.

[137] Hermann Weyl. *Symmetry*. Princeton Science Library, 1989.

[138] Stephen Wolfram. *A New Kind of Science*. Wolfram Media, Inc., 2002.

[139] Michael T. Wong, Douglas E. Zongker, and David H. Salesin. Computer–generated floral ornament. *Proceedings of SIGGRAPH'98*, pages 423–434, 1998.

[140] Jane Yen and Carlo Séquin. Escher sphere construction kit. In *Proceedings of the 2001 symposium on Interactive 3D graphics*, pages 95–98. ACM Press, 2001.

[141] Douglas Zongker. Celtic knot thingy. `http://www.cs.washington.edu/homes/dougz/hacks/knot/`.

[142] Douglas Zongker. Creation of overlay tilings through dualization of regular networks. In Nathaniel Friedman and Javiar Barrallo, editors, *ISAMA 99 Proceedings*, pages 495–502, 1999.

VITA

Craig Steven Kaplan was born in 1972 in Montréal, Québec. As a child, he would spend hours staring at an image of Escher's *Relativity* in a book.

Craig spent five years as an undergraduate at the University of Waterloo, studying pure mathematics and computer science. He also spent a number of internships at Alias|Wavefront in Toronto. He then ventured to the beautiful Pacific Northwest to pursue a PhD at the University of Washington in Seattle. At the end of 2002 Craig, his wife Nathalie, their new daughter Zoë, and their three cats will make the trek back to Ontario, where he will return to Waterloo's School of Computer Science, this time as a member of the faculty.