Computer Generated Islamic Star Patterns

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Abstract

Islamic star patterns are a beautiful and highly geometric art form. Many analyses have been done of their complex structure and symmetries. We pick up one line of analysis based on placing stars and rosettes in a formation dictated by a tiling of the plane, and develop a software implementation of the technique. We discuss the construction of the designs and show some computer-generated results.

1 Introduction

One of the most famous and most beautiful forms of geometric art is the Islamic star pattern. Mathematically, an Islamic star pattern is a planar arrangement of line segments that together delineate copies of a small number of different shapes, some of which are stars. These designs have been used as decoration in Islamic cultures for over a thousand years, and have still not lost their appeal.

Aside from the intuitive appreciation of their beauty, historians and mathematicians are interested in Islamic star patterns because they represent an interesting puzzle. The theory behind the patterns was a closely-guarded secret of the craftsmen who designed them. As a result, little information (other than the finished works of art) survives about the techniques used to create the patterns.

From the many attempts to recreate existing Islamic patterns and generate new ones, we now have a variety of useful analyses and constructions. Grünbaum and Shephard [8] decompose periodic Islamic patterns by their symmetry groups, obtaining a fundamental region they use to derive properties of the original pattern. Abas and Salman [1] carry this process out on a vast collection of patterns from every periodic symmetry group. Dewdney [4] proposes a method of reflecting lines off of periodically-placed circles. Castera [6] presents a technique based on the construction of networks of eightfold stars and “safts”.

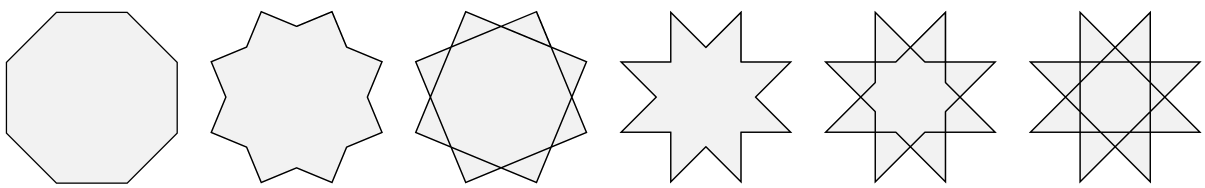
This paper presents a technique described by E.H. Hankin [9] based on his experiences seeing partially-finished installations of Islamic art. It also incorporates the work of A.J. Lee [10], who built upon Hankin’s findings by providing simple constructions for the common features of Islamic patterns. The technique has been developed into a software implementation that was used to produce the patterns in this paper. The implementation is available as a Java applet for experimentation at `http://www.cs.washington.edu/homes/csk/tile/islam/`.

The rest of the paper proceeds as follows. Section 2 presents constructions for the common features of Islamic patterns: stars and rosettes. Section 3 shows how complete designs may be built using repeated copies of those features. Techniques for creating visually appealing renderings of the designs are given in section 4. Some results appear in section 5. The paper concludes in section 6 by exploring some opportunities for future work.

2 Stars and Rosettes

An n-pointed star polygon can be drawn by distributing n points evenly along the perimeter of a circle, choosing a number d, and connecting each point to the dth point encountered after it on the circle. The resulting shape is commonly referred to as (n/d), and is defined when $2 < d < \frac{n}{2}$. When $d = 1$, we obtain the usual regular n-gon.

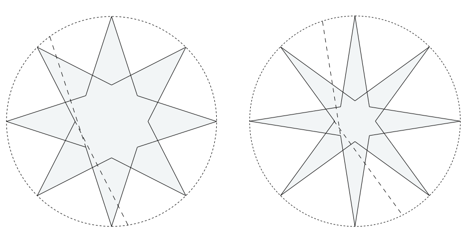
Stars that appear in Islamic ornament are often missing some of the line segments of (n/d). We extend our star notation with a value s that indicates how much of each edge of the star to draw in terms of the edge's intersections with other edges. An $(n/d)s$ star is an n-pointed star, connected in jumps of d points, drawing each edge up to its sth intersection with other edges. Figure 1 shows the different stars that are possible when $n = 8$.



$(8/1)1$ $(8/2)1$ $(8/2)2$ $(8/3)1$ $(8/3)2$ $(8/3)3$

Figure 1: The six possible eight-pointed stars.

It is also possible to give an interpretation of $(n/d)s$ when d is not an integer, by taking d to mean "move $\frac{d}{n}$ of the way around the circle". In order to preserve the star's symmetry, the edge develops a bend at its midpoint and bends back to connect to the circle $\frac{d-\lfloor d \rfloor}{n}$ of the way from the original vertex. When d is an integer, this extended definition gives the same star as above.



$(8/3.1)2$ $(8/3.6)2$

Figure 2: $(n/d)s$ stars when d is not an integer.

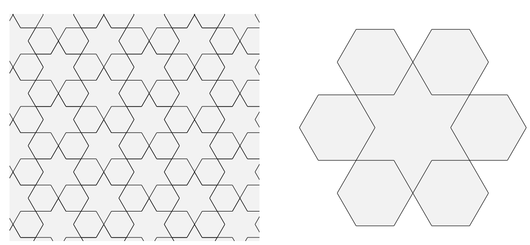


Figure 3: An arrangement of 6-stars can be reinterpreted as rosettes.

When sixfold stars are arranged as in figure 3, a higher-level structure emerges: every star is surrounded by a ring of regular hexagons. The pattern can be regarded as being composed of these surrounded stars, or "rosettes". Placing copies of the rosette in the plane will leave behind gaps, which in this case happen to be more sixfold stars. This arrangement is one of the oldest in the Islamic tradition.

The rosette, a central star surrounded by hexagons, appears frequently in Islamic art, in the six-pointed form shown above, but also with eight, nine, ten, twelve and many other numbers of points. Since the outer hexagons cannot be regular when n is not six, we need to generalize the construction of rosettes for any number of points. A.J. Lee [10] shows how to construct an n-fold rosette inside a regular n-gon in such a way that the points of the rosette (the tips of the outer hexagons) lie on the midpoints of the edges of the n-gon.

Lee's rosette construction, shown in figure 4, finds a rosette with highly regular outer hexagons in the sense that the four outer edges have equal lengths, $\angle AFE = \angle ADE$, and the sides of the hexagons are parallel. The key point E is the intersection of $\overline{CD}$ and the bisector of $\angle OAB$. The side of the hexagon becomes the line through E parallel to the radius $\overline{OD}$. This line intersects radius $\overline{OA}$ at F, which becomes one of the points of the rosette's inner star. The remaining points of the rosette follow from symmetry and from the construction of the inner star, as above. The rosette can be varied continuously by sliding E along the bisector of $\angle AOB$.

Some Islamic designs feature a motif slightly more complicated than a basic rosette, where opposing limiting edges from adjacent tips of the rosette are joined up. The resulting object has the same symmetries and number of outer points as the rosette, but with an additional layer of geometry on its outside. We refer to these as "extended rosettes". Figure 5 shows a ninefold extended rosette.

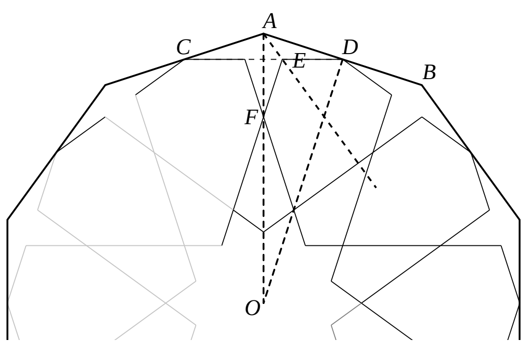


Figure 4: The construction of a ten-pointed rosette.

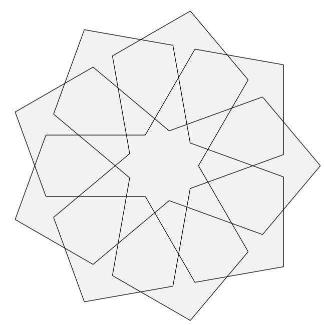


Figure 5: An extended rosette.

3 Filling the Plane

Equipped with a variety of typically Islamic motifs that can be inscribed in regular polygons, we are now ready to create complete periodic designs. We start with a periodic tiling containing regular polygons, with irregular polygons thrown in as needed to fill gaps. For each regular n-gon, we choose a n-fold star, rosette or extended rosette to place in it and replicate that motif everywhere the n-gon appears in the tiling.

The result is a design like that of figure 6(b). There are still large gaps where motifs were not placed. In figure 6, the gaps correspond to the squares in the original tiling. Each square edge is shared with an edge of an octagon, and therefore there is rosette tip incident to it. The presence of these tips suggests a technique for filling the gaps in a natural way, by extending the line segments that terminate on the boundary of the gap until they meet other extended segments inside the gap. Following this procedure guarantees that the resulting design will admit an interlacing.

Figure 6(c) shows the design with the free rosette tips extended into the gaps. Here, the natural extension creates regular octagons in the interstitial regions. To complete the construction, the original tiling is removed, resulting in the design in figure 6(d), a well-known Islamic star pattern.[2, plate 48]

Given a tiling containing regular polygons and possibly irregular gaps, we can now construct a wide range of different designs by starting with different motifs. Even when restricted to the octagon-square

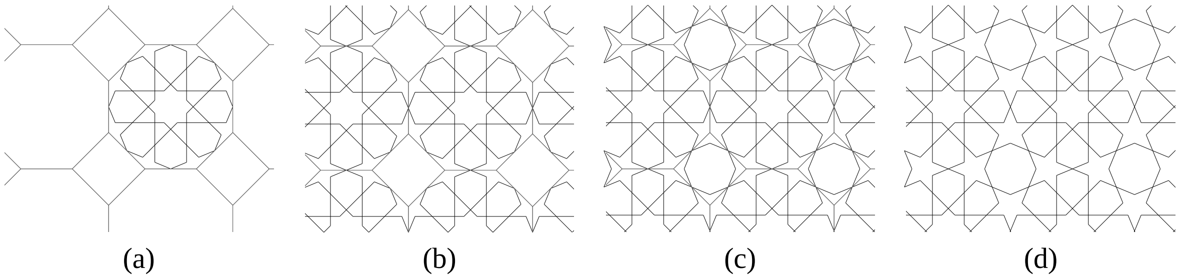


Figure 6: Given the octagon and square tiling shown in (a), we decide to place 8-fold rosettes in the octagons and let the system infer geometry for the squares. The rosette is copied to all octagons in (b), and lines from unattached tips are extended into the interstitial spaces until they meet in (c). The construction lines are removed, resulting in the final design shown in (d).

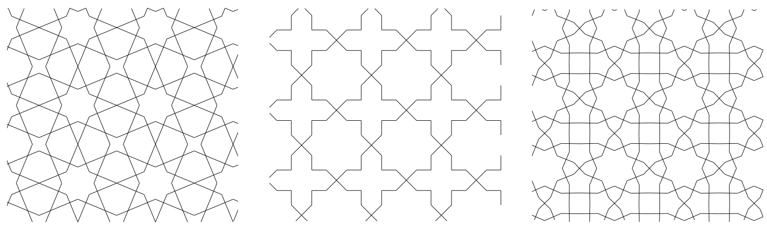


Figure 7: Some alternative patterns based on the octagon-square tiling, that can be constructed by varying the motif placed in the octagons.

tiling used above, many different designs can be created by choosing different motifs for the octagons. Three alternative designs appear in figure 7. Of course, we can expand the range of this technique in the other dimension by encoding a large number of different tilings.

The implementation currently encodes a dozen tilings from which Islamic star patterns may be produced. Some are familiar regular or Archimedean tilings.[7, sec. 2.1] Some are derived by examination of well-known Islamic patterns. The remaining tilings were discovered by experimentation and lead to novel Islamic designs shown in section 5.

4 Rendering

The output of the construction process is a planar graph. To be sure, the graph has an intrinsic beauty that holds up when it is rendered as simple line art. Historically, however, these designs were never merely drawn as lines. Islamic star patterns are typically used as a decoration for walls and floors. The faces of the planar graph are realized as a mosaic small terracotta tiles in a style known as "Zellij". Often, the edges are thickened and incorporated into the mosaic with narrow tiles, sometimes broken up to suggest an interlacing pattern. Islamic designs can also be found carved into wood or stone and built into trellises and latticework. To increase the aesthetic appeal of our implementation, we provide the ability to render the planar graph in a manner reminiscent of some of these techniques.

The simplest nontrivial rendering technique is to thicken the edges of the planar graph. This "outline" rendering style adds weight character to lines of the plain style.

The "emboss" style simulates the appearance of a wooden trellis. The edges are treated as wooden strips with square cross-section, rotated so that a vertex of the square points towards the viewer. The strips

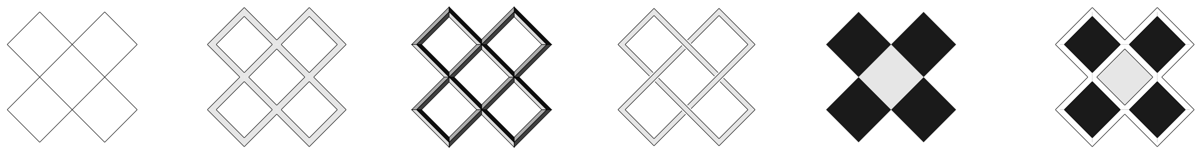


Figure 8: Rendering styles.

are then rendered by specifying a direction for a fictitious light source that gives the final drawing a three-dimensional look.

The "interlace" style starts with the outline style and adds line segments at each crossing in the graph to suggest an over-under relationship between the crossing edges. When every vertex in the graph has degree two or four, the crossings can always be chosen so that the graph is broken into long strands that adhere to a strict alternation of over and under in their intersections with other strands.

The final style, ”checkerboard”, renders the faces of the graph and not the edges. When all vertices have even degree (as they must in an interlace design), it is always possible to colour the faces with only two colours in such a way that faces with the same colour never share an edge. The checkerboard style walks the graph, creating a consistent 2-colouring.

A further enhancement can be achieved by layering one of the edge-based rendering styles on top of the checkerboard style. The overall picture perhaps comes closest in appearance to Zellij.

5 Results

The two final pages of this paper present a selection of finished computer-generated drawings. The first group, figure 9, is made up of reproductions of well-known Islamic star patterns which can be found in Bourgoin [2] or Abas and Salman [1]. Figure 10 contains designs that do not appear in either of those sources. Three of them are based on polygonal tilings that do not seem to be used by any known designs.

6 Future Work

The software implementation and the technique on which it is based allow access to a wide space of designs without offering so much flexibility that it becomes overly easy to wander out of the space of recognizably Islamic patterns. There are, however, opportunities for future work that do not compromise the focus of the system.

The set of available tilings from which to form patterns is open-ended. More tilings could be implemented. Some new ones can easily be derived by inspection of patterns in Bourgoin or Abas and Salman. We could move away from periodicity by implementing aperiodic tilings with regular polygons. At last year's ISAMA conference, Castera demonstrated an aperiodic Islamic star pattern based on Penrose rhombs. [3] Finally, the hyperbolic plane offers tremendous freedom in the construction of tilings with regular polygons. We hope to adapt the technique described in this paper to the Poincaré model of the hyperbolic plane, much the same way Dunham has done with Escher patterns. [5]

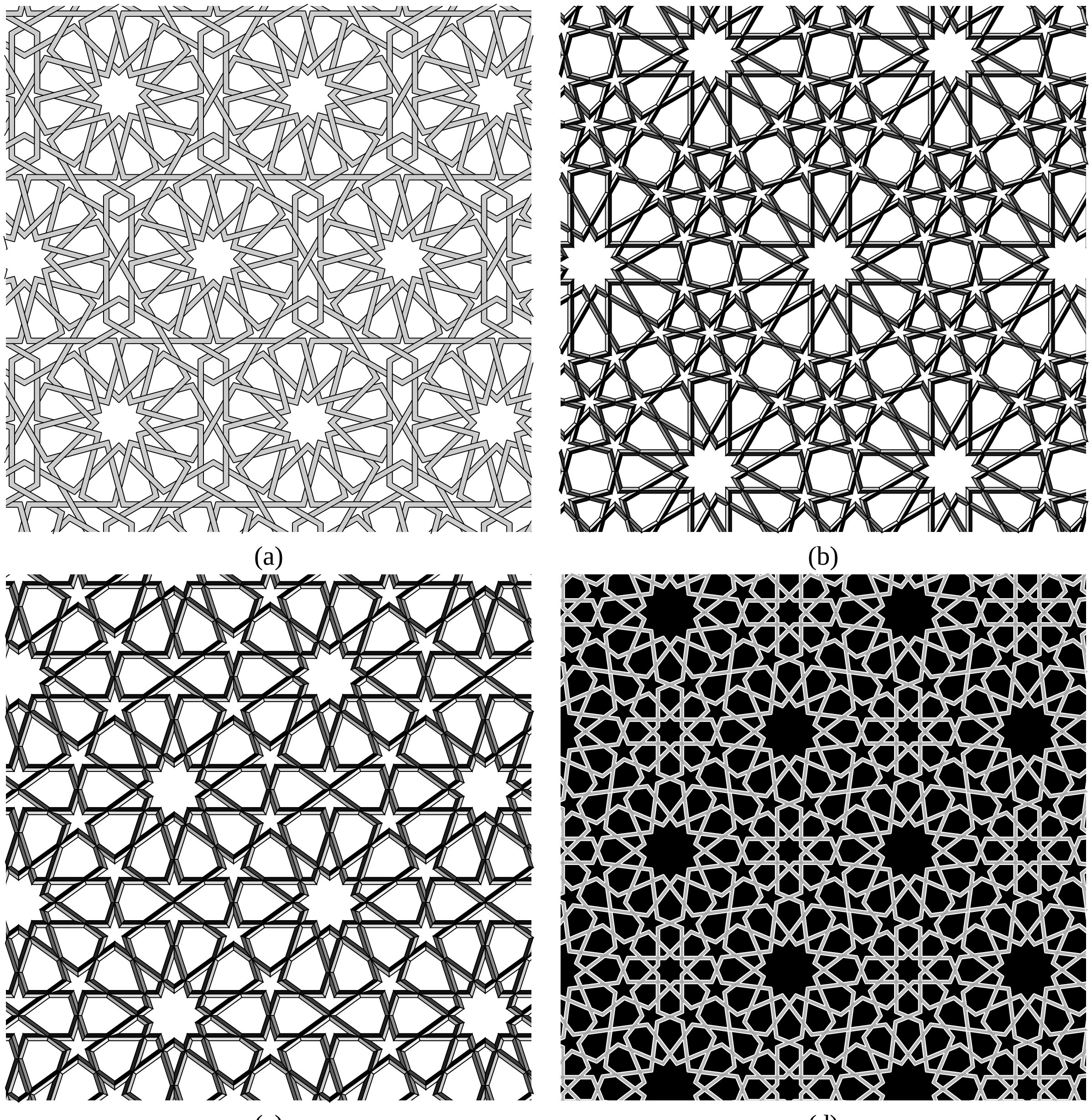


Figure 9: Results.

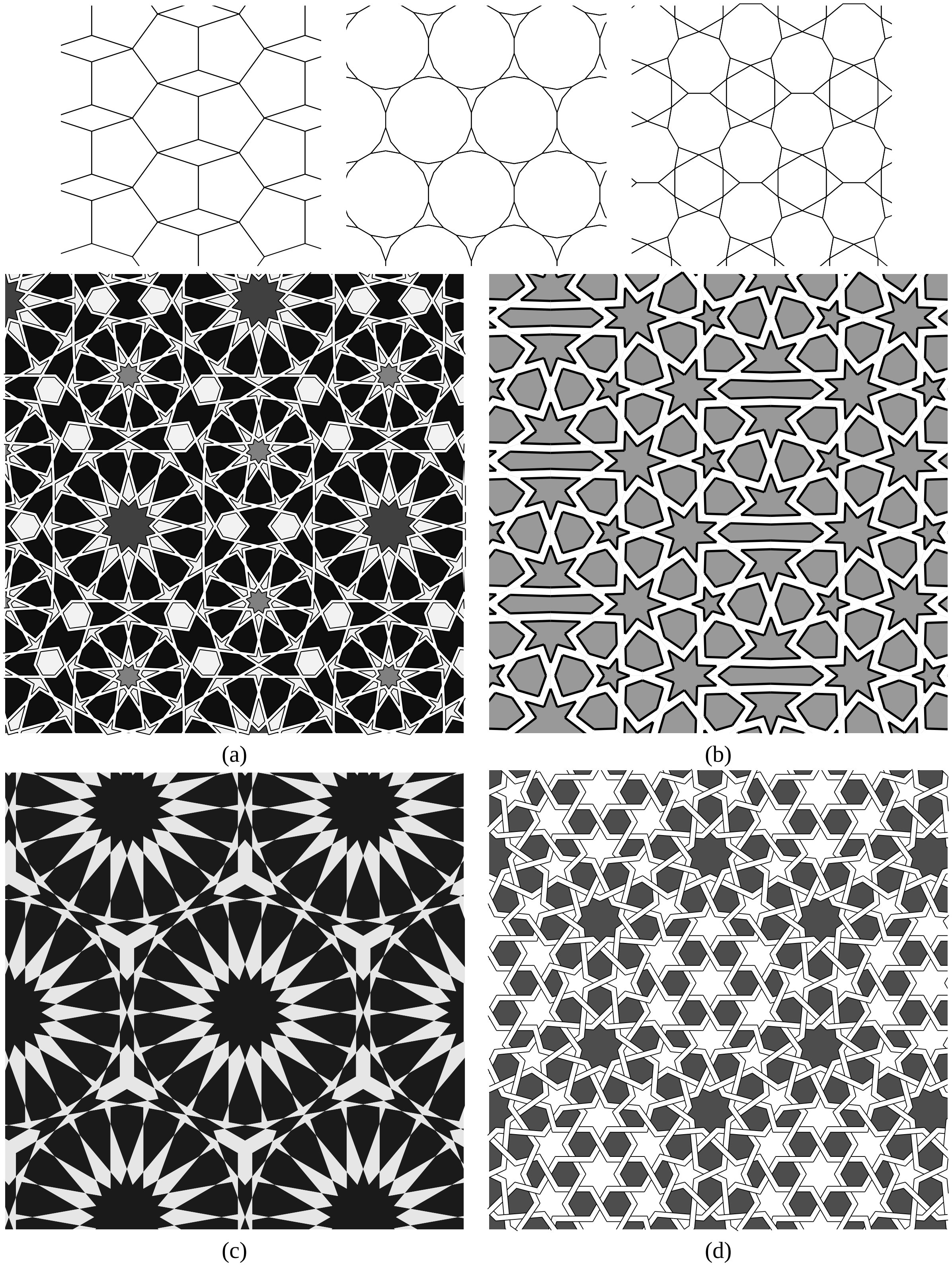


Figure 10: Finished star patterns not found in the literature. The pattern in (a) is similar to one found in Abas and Salman [1, p. 93], using extended rosettes instead of ordinary rosettes. The other three patterns are based on previously unused polygonal tilings, shown in the top row.

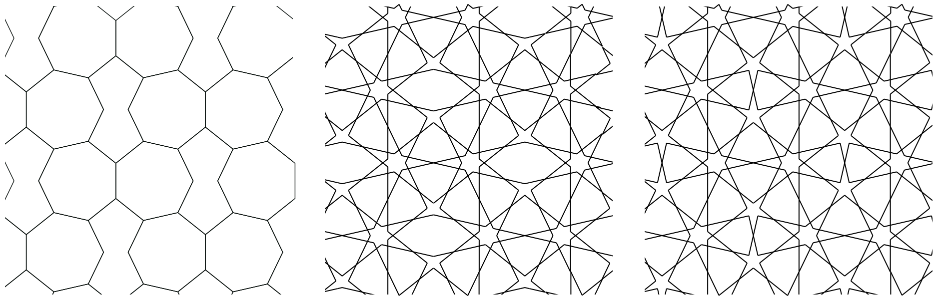


Figure 11: A novel pattern with 7-stars. The design in the centre, resulting from the natural extension of star edges, leaves behind disproportionate octagons. The design on the right, constructed manually, corrects this by redistributing the area to new 5-stars.

One last aspect of the system we hope to improve is the naïve extension of lines into interstitial regions. Our approach can easily fail to produce attractive results. In figure 11, a novel grid based on regular heptagons is turned into an Islamic pattern by placing $(7/3)2$ stars in the heptagons. The natural extension of star edges into the gaps leaves large, unattractive octagonal areas. We can improve the final design by breaking up the octagon, creating two 5-stars and some leftover rosette-like hexagons. At present, we have not devised an algorithm that can automatically determine when and where to add these breaks.

Acknowledgments

I am indebted to Mamoun Sakkal for his guidance and helpful discussion while initiating this project in his course. Thanks also to David Salesin for his valuable input and to Reza Sarhangi for the encouragement.

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