Matrix-nullspace Wavelet Construction

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Abstract. We discuss a method for constructing (pre-)wavelets by finding the nullspace of a matrix whose elements are generated from scale functions and their scale relation. In particular, we show how the size of this matrix may be known in advance and give examples of its construction.

§Introduction

We consider the usual setting of a *multiresolution analysis* [3] in which a basis of *scale functions* $\phi_{j,m}(x) = \phi(\sigma^j x - m)$ (assumed of compact support: $\phi(x) = 0$ when $x \notin [0, M]$) generates a space $\mathcal{V}^j$, *dilated* scale functions $\phi_{j+1,m}(x) = \phi(\sigma^{j+1} x - m)$ generate a space $\mathcal{V}^{j+1}$, and a *scale relation* $\phi(x) = \sum_m p_m \phi(\sigma x - m)$ (assumed to be a finite sum: only $p_0, \ldots, p_s \neq 0$) provides a nesting of spaces $\cdots \mathcal{V}^j \subset \mathcal{V}^{j+1} \cdots$. Our goal is to construct a basis of *wavelet functions* $\psi_{j,n}(x) = \psi(\sigma^j x - n)$ that span $\mathcal{W}^j$, the orthogonal complement of $\mathcal{V}^j$ in $\mathcal{V}^{j+1}$ with respect to a given inner product $\langle \cdot, \cdot \rangle$, by providing a representation $\psi(x) = \sum_n q_n \phi(\sigma x - n)$ (also finite: only $q_0, \ldots, q_t \neq 0$) with $\langle \psi(x-k), \phi(x-\ell) \rangle = 0$ for all k,ℓ.

From the point of view of the material presented in [4], we shall be completing the basis for $\mathcal{V}^0$ to one for $\mathcal{V}^1$ by the straightforward matrix technique of *QL decomposition*, using the matrix structure to predict the number of summands needed for ψ. The scale relation is used in the obvious way to extend the completion to general j.

The beginnings of this matrix approach have appeared in [5]. We add to the material in that paper a strategy for determining *a priori* how many coefficients q_n will be needed for ψ. We also mention situations in which a finite number of special scale functions exist in addition to the regular ones, as in [7], in which bivariate scale functions (e.g. box splines [1]) can be folded into the matrix setting, and in which the inner product can be changed from an integral to a discrete sum.

This paper is intended only to provide a quick scheme for creating the functions ψ. It requires no more knowledge of ϕ than the scale relation and no more involvement with ϕ than the ability to compute the inner products

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$\langle \phi(\sigma x - k), \phi(\sigma x - r) \rangle$. One must be assured from other considerations that the result will provide a stable basis suitable for one's application. Finally, since the construction provides the q_n as the components of a vector in a matrix nullspace, any multiple of this vector will also satisfy the construction. This means that the normalization of the functions ψ also needs to be obtained from other considerations.

When other considerations support the success of the construction, the functions $\phi_{j,m}$, $\psi_{j,n}$ provide a basis for $\mathcal{V}^j \oplus \mathcal{W}^j$, and the functions $\phi_{j+1,m}$ provide a basis for the same space in its unstructured form $\mathcal{V}^{j+1}$. The coefficients p_m, q_n are the essential elements describing the change of basis $\mathcal{V}^{j+1} \to \mathcal{V}^j \oplus \mathcal{W}^j$. Coefficients a_i, b_i satisfying $\phi(\sigma x - \ell) = \sum_r [a_{\ell-\sigma r}\phi(x-r)$ $+b_{\ell-\sigma r}\psi(x-r)]$ provide the change of basis in the opposite direction. Matrix methods for determining a_i, b_i are given in [7]. The use of a_i, b_i for the *decomposition* of data into levels of detail, and of p_m, q_n for the *reconstruction* of the data from those levels of detail, appear in numerous references.

§Wavelet Construction

We shall not assume that we know the number of nonzero q_n or the support of the ψ as we begin the construction, but the assumption that the sequence $\{p_m\}$ is finite and the supports of the functions $\phi_{j,m}$ are compact will ensure that a finite sequence $\{q_n\}$ can be found and that the supports of the $\psi_{j,n}$ will be compact. Consequently, $\phi_{j,k}$ and $\psi_{j,\ell}$ will be orthogonal for most k and ℓ, since their supports do not intersect. The goal is to determine finite sequences $\{q_n\}$ so that $\langle \phi_{j,k}, \psi_{j,\ell} \rangle = 0$ for the remaining k and ℓ.

Because of the derivation of all $\phi_{j,m}(x)$,$\psi_{j,n}(x)$ as scales and shifts of $\phi(x)$,$\psi(x)$ respectively, it is enough to obtain $\langle \psi(x), \phi(x-\ell) \rangle = 0$ for a finite sequence of values for ℓ. Using the scale relation:

$$\begin{aligned} 0 &= \langle \psi(x), \phi(x-\ell) \rangle \\ &= \left\langle \sum_k q_k \phi(\sigma x - k), \sum_r p_{r-\sigma\ell} \phi(\sigma x - r) \right\rangle \\ &= \sum_k q_k \left[\sum_r p_{r-\sigma\ell} \langle \phi(\sigma x - k), \phi(\sigma x - r) \rangle \right] \end{aligned}$$

which can be viewed as a matrix nullspace problem $\mathrm{q}^T \mathrm{P} = 0$ where

$$\mathrm{P} = \; k \begin{pmatrix} \cdots & \overset{r}{\cdots} & \cdots \\ \cdots & \langle \phi(\sigma x - k), \phi(\sigma x - r) \rangle & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} r \begin{pmatrix} \cdots & \overset{\ell}{\cdots} & \cdots \\ \cdots & p_{r-\sigma\ell} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

The structure of each columns of P is indicated schematically in Figure 1. The rows correspond to the *individual* pairwise products $q_k\phi(\sigma x - k)$, where we know that the first row index of importance is $k = 0$, but we do not yet

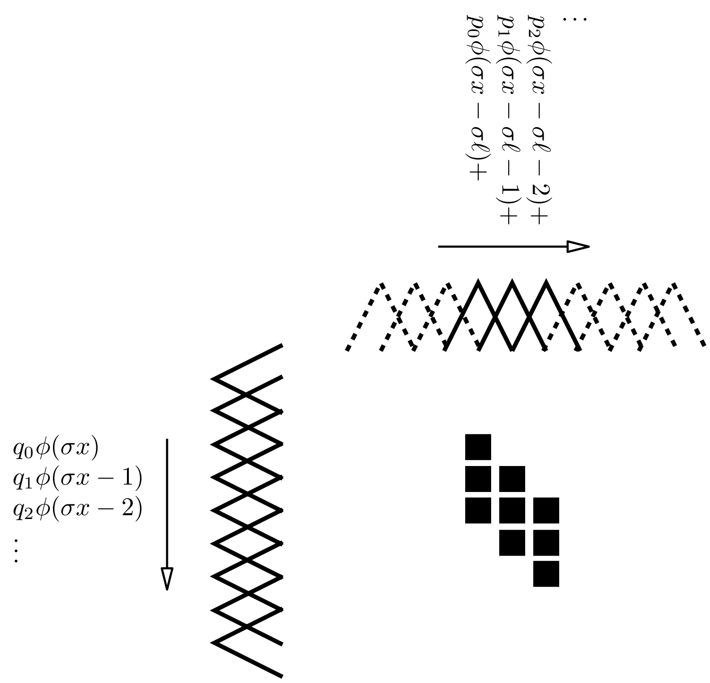

Fig. 1. Structure leading to column ℓ of P.

know the last row index. The ℓ-th column will correspond to the *sum* of the pairwise products $p_r\phi(\sigma x - \sigma\ell - r)$, where the first index ℓ of importance will be the least integer for which the support $[0, N]$ of $\psi(x)$ and the support $[\ell, M + \ell]$ of $\phi(x - \ell)$ overlap; namely, $\ell = 1 - M$. The bottom portion of Figure 1 indicates the overlap for $\sigma = 2$, $M = 2$, and $p_0, p_1, p_2 \neq 0$, for example the case when ϕ is the cardinal linear B-spline. All shifts $\phi(\sigma x - \cdot)$ are shown horizontally and vertically, with the horizontal shifts shown dashed corresponding to zero coefficients p_m. The black boxes indicate pairs of shifts whose supports overlap, producing a nonzero inner product. The ℓ-th column will contain the vector sum of the values indicated by the black boxes, resulting in 5 nonzero entries for the given figure.

Matrix P constitutes a slice of an infinite matrix. The row index $k = 0$ and the column index $\ell = 1 - M$ provide the left and upper boundaries of the slice respectively. Each column of the infinite matrix shifts σ rows down from the preceding column. Figure 2 gives a general indication of how many nonzero entries will appear in each column; namely $s + 2M - 1$, the number of shifts to which $\phi(\sigma x - \cdot)$ can be subjected (including the zero shift) and still intercept $\sum_{m=0}^{s} p_m\phi(\sigma x - m)$.

§The Matrix Size

A naive approach for generating the sequence $\{q_n\}$ might be as follows. For each $t = 0, 1, \ldots$ construct the version of P having $t + 1$ rows and t columns (which ensures a one-dimensional nullspace). Generate the nullspace vector $q = (q_0\,, \ldots, q_t)$, and stop when this q is in the nullspace of all larger versions of P. Figure 3 provides a view of the zero/nonzero structure of such a sequence

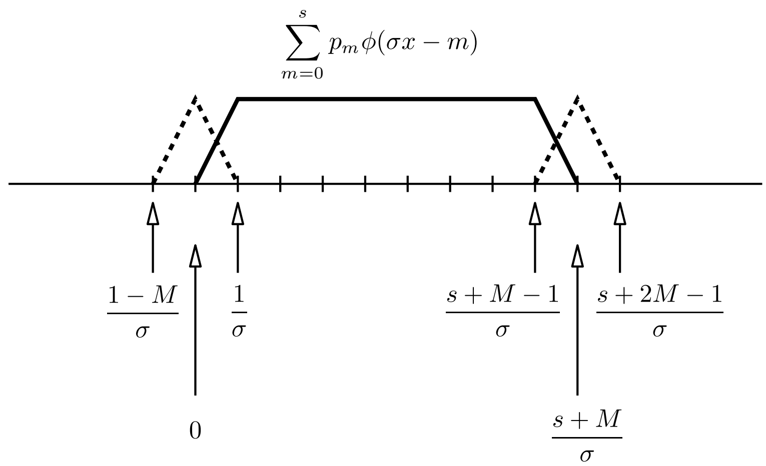

Fig. 2. Overlap of support.

$$t = 1, 2, 3 \qquad \begin{bmatrix} p \\ p \end{bmatrix} \quad \begin{bmatrix} p & p \\ p & p \\ 0 & p \end{bmatrix} \quad \begin{bmatrix} p & p & 0 \\ p & p & p \\ 0 & p & p \\ 0 & p & p \end{bmatrix}$$

$$t = 4, 5 \qquad \begin{bmatrix} p & p & 0 & 0 \\ p & p & p & 0 \\ 0 & p & p & 0 \\ 0 & p & p & p \\ 0 & 0 & p & p \end{bmatrix} \quad \begin{bmatrix} p & p & 0 & 0 & 0 \\ p & p & p & 0 & 0 \\ 0 & p & p & 0 & 0 \\ 0 & p & p & p & 0 \\ 0 & 0 & p & p & 0 \\ 0 & 0 & p & p & p \end{bmatrix}$$

Fig. 3. Successive matrix slices.

for the case of cardinal linear B-splines.

In general, the first column of each slice will begin with σ nonzero elements (the number of intervals of overlap between the support of $\phi(x+M-1)$ and the successive shifts of $\phi(\sigma x - k)$) – or a portion thereof, if the slice has fewer than σ rows. Each successive column will shift down by σ rows, and each column will have $s + 2M - 1$ nonzero entries. Thus, all versions of P intercept the columns of the infinite matrix at their upper left corner in the way indicated by Figure 4. From this it is easy to determine the smallest value of t for which the rightmost column of of the $(t+1) \times t$ matrix slice P will have zeros in all save possibly its bottom position: $t \geq \frac{s+2M-1}{\sigma-1}$. This is the case for $t = 5$ in Figure 3 ($\sigma = 2$, $s = 2$, $M = 2$ – the cardinal linear B-splines).

§Determining the Nullspace

The decomposition of a matrix into the product of an orthogonal matrix and a matrix whose nonzero elements are arranged in a triangular pattern is covered in [6]. In the present case we are interested in the *QL* factorization

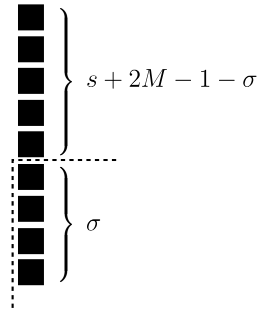

Fig. 4. First column of a slice.

of a $(t+1) \times t$ matrix $\mathrm{P} = \mathrm{Q}\begin{bmatrix} 0 \\ \mathrm{L} \end{bmatrix}$ as indicated:

$$\begin{bmatrix} \mathrm{p} & \mathrm{p} & 0 \\ \mathrm{p} & \mathrm{p} & 0 \\ \mathrm{p} & \mathrm{p} & 0 \\ \mathrm{p} & \mathrm{p} & \mathrm{p} \end{bmatrix} = \begin{bmatrix} \mathrm{q} & \mathrm{q} & \mathrm{q} & 0 \\ \mathrm{q} & \mathrm{q} & \mathrm{q} & 0 \\ \mathrm{q} & \mathrm{q} & \mathrm{q} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ \ell & 0 & 0 \\ \ell & \ell & 0 \\ \mathrm{p} & \mathrm{p} & \mathrm{p} \end{bmatrix}$$

The importance of the zeros in the last column of P become clear. Due to these zeros, the matrix Q can be structured so that its last row and column are the corresponding parts of the identity matrix. The leftmost column of Q provides a vector in the nullspace of P, and each lengthened version of this column will, by virtue of its zero structure, be in the nullspace of each larger version of P.

For numerical reasons it is better to work with symbolic algebra than to perform the calculations of a *QL* decomposition. The *Maple* [2] system has a command to produce the nullspace of a matrix, and we have used it with success.

§Three Simple Examples

For cardinal linear B-splines, the example that has served as an illustration, we have

$$p_0 = \frac{1}{2},\ p_1 = 1,\ p_2 = \frac{1}{2}$$

$$\mathrm{P} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{24} & \frac{5}{12} & \frac{1}{24} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{24} & \frac{5}{12} & \frac{1}{24} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{24} & \frac{5}{12} & \frac{1}{24} \end{bmatrix}$$

for which *Maple* reports the following nullspace vector:

$$q_0 = 1,\ q_1 = -6,\ q_2 = 10,\ q_3 = -6,\ q_4 = 1$$

Within a scale factor, this nullspace vector reproduces the minimum-support cardinal linear B-spline wavelet.

Bézier wavelets offer a variation on this theme. Each single scale relation now splits into several. Since the nullspace of a matrix is independent of the ordering of the columns, the scale relations can be grouped into adjoining columns, and the previous discussion of columnwise structure can be reinterpreted for blocks of columns. A simple illustration can be given by the quadratic case, in which the columns come in blocks of three, one for each of the scale functions $\phi_i(x) = \binom{2}{i}(1-x)^{(2-i)}x^i$. In fact, since intervals of support are disjoint; the matrix P decouples into independent blocks, and only one block needs to be dealt with:

$$\begin{array}{lcllllllll}
\phi_0(x) & = & \phi_0(2x) & + & \frac{1}{2}\phi_1(2x) & + & \frac{1}{4}\phi_2(2x) & + & \frac{1}{4}\phi_0(2x-1) \\
\phi_1(x) & = & \frac{1}{2}\phi_1(2x) & + & \frac{1}{2}\phi_2(2x) & + & \frac{1}{2}\phi_0(2x-1) & + & \frac{1}{2}\phi_1(2x-1) \\
\phi_2(x) & = & \frac{1}{4}\phi_2(2x) & + & \frac{1}{4}\phi_0(2x-1) & + & \frac{1}{2}\phi_1(2x-1) & + & \phi_2(2x-1)
\end{array}$$

$$\begin{bmatrix}
\frac{31}{240} & \frac{1}{30} & \frac{1}{240} \\
\frac{23}{240} & \frac{7}{120} & \frac{1}{80} \\
\frac{1}{15} & \frac{3}{40} & \frac{1}{40} \\
\frac{1}{40} & \frac{3}{40} & \frac{1}{15} \\
\frac{1}{80} & \frac{7}{120} & \frac{23}{240} \\
\frac{1}{240} & \frac{1}{30} & \frac{31}{240}
\end{bmatrix}$$

The dimensions of P (6×3) allow for three, linearly independent nullspace vectors, which *Maple* can provide, yielding the wavelets:

$$\begin{array}{lcrcrcrcl}
\psi_0(x) & = & -8\phi_0(2x) & + & 19\phi_1(2x) & - & 12\phi_2(2x) & + & 1\phi_1(2x-1) \\
\psi_1(x) & = & -13\phi_0(2x) & + & 30\phi_1(2x) & - & 18\phi_2(2x) & + & 1\phi_2(2x-1) \\
\psi_2(x) & = & -4\phi_0(2x) & + & 10\phi_1(2x) & - & 7\phi_2(2x) & + & 1\phi_0(2x-1)
\end{array}$$

The same approach can be used to provide coefficients for other wavelets on an interval; for example, taking the quadratic B-splines with knots $\{0, 0, 0, 1/8, \ldots, 7/8, 1, 1, 1\}$ as coarse-scale functions and those with knots $\{0, 0, 0, 1/16, \ldots, 15/16, 1, 1, 1\}$ as fine-scale functions, then the nullspace vectors shown in Figure 5 provide coefficients for wavelets. (These vectors are symmetrized versions of those produced by *Maple* so that they progress from left to right and right to left identically.)

§Box Splines

As an experiment in extending this approach, we have investigated constructing 3-direction, bivariate box-spline wavelets for scale relations $\sigma = 2, 4$. The nullspace construction continues to work, but with a few details of complication. The lattice structure of the univariate domain made translation to a matrix easy: indices k, ℓ for shifts $\phi(\sigma x - k), \phi(\sigma x - \ell)$ could be allocated to

$$(\frac{286}{945},-\frac{1391}{3780},\frac{4967}{22680},-\frac{2083}{22680},\frac{29}{1620},-\frac{1}{1620},0,0,0,0,0,0,0,0,0,0,0,0)$$
$$(0,\frac{13}{192},-\frac{1949}{11520},\frac{3481}{11520},-\frac{1681}{5760},\frac{809}{5760},-\frac{319}{11520},\frac{11}{11520},0,0,0,0,0,0,0,0,0,0)$$
$$(0,0,-\frac{1}{960},\frac{29}{960},-\frac{49}{320},\frac{101}{320},-\frac{101}{320},\frac{49}{320},-\frac{29}{960},\frac{1}{960},0,0,0,0,0,0,0,0)$$
$$(0,0,0,0,-\frac{1}{960},\frac{29}{960},-\frac{49}{320},\frac{101}{320},-\frac{101}{320},\frac{49}{320},-\frac{29}{960},\frac{1}{960},0,0,0,0,0,0)$$
$$(0,0,0,0,0,0,\frac{1}{960},-\frac{29}{960},\frac{49}{320},-\frac{101}{320},\frac{101}{320},-\frac{49}{320},\frac{29}{960},-\frac{1}{960},0,0,0,0)$$
$$(0,0,0,0,0,0,0,0,\frac{1}{960},-\frac{29}{960},\frac{49}{320},-\frac{101}{320},\frac{101}{320},-\frac{49}{320},\frac{29}{960},-\frac{1}{960},0,0)$$
$$(0,0,0,0,0,0,0,0,0,0,\frac{11}{11520},-\frac{319}{11520},\frac{809}{5760},-\frac{1681}{5760},\frac{3481}{11520},-\frac{1949}{11520},\frac{13}{192},0)$$
$$(0,0,0,0,0,0,0,0,0,0,0,0,-\frac{1}{1620},\frac{29}{1620},-\frac{2083}{22680},\frac{4967}{22680},-\frac{1391}{3780},\frac{286}{945})$$

Fig. 5. Example for quadratic B-splines on an interval..

successive rows and columns of P, and since support overlap was the same as interval intersection, the nonzero structure of a column was easy to determine. For the box splines under consideration we have two indices, ℓ_1, ℓ_2, arising from the scale relation and two more, k_1, k_2, arising from the wavelet representation:

$$\begin{aligned} \phi_\Xi(x_1,x_2) &= \textstyle\sum_{\ell_1,\ell_2} p_{\ell_1,\ell_2}\phi_\Xi(\sigma x_1-\ell_1,\sigma x_2-\ell_2) \\ \psi_\Xi(x_1,x_2) &= \textstyle\sum_{k_1,k_2} q_{k_1,k_2}\phi_\Xi(\sigma x_1-k_1,\sigma x_2-k_2) \end{aligned}$$

where Ξ indicates the matrix, $\Xi = (\xi^1 : \mu_1, \xi^2 : \mu_2, \xi^3 : \mu_3)$, of directions ξ^i with multiplicities μ_i. We must map the pairs (k_1, k_2) into the rows of the matrix and the pairs (ℓ_1, ℓ_2) into the columns, and we must determine at which row-column intersections the support of $\phi_\Xi(\sigma x_1 - k_1, \sigma x_2 - k_2)$ will overlap the support of

$$\phi_\Xi(x_1-\ell_1,x_2-\ell_2) = \sum_{r_1,r_2} p_{r_1-\sigma\ell_1,r_2-\sigma\ell_2}\phi_\Xi(\sigma x_1-r_1,\sigma x_2-r_2)$$

This situation is shown in Figure 6, where the support of $\phi_\Xi(x_1, x_2)$ is on the lower left and the support of a shift of $\phi_\Xi(\sigma x_1, \sigma x_2)$ is shown on the upper right. The "home position" ($x_1 = x_2 = 0$) of each domain is indicated by a dot. The domains are hexagonal (reducing to rectangular in the tensor-product case), and the dimensions $D_1, \ldots, d_3$ are determinable from Ξ and the degree [1]. An option that works involves ordering the columns first, starting with the lower bound $(L_1, L_2) = (\ell_1, \ell_2) = (-D_1 - D_3, -D_2, -D_3)$ and numbering the columns according to the consecutive terms in the diagonal sequence

$$\begin{array}{lll} (L_1,L_2)+(1,0), & (L_1,L_2)+(0,1), & \\ (L_1,L_2)+(2,0), & (L_1,L_2)+(1,1), & (L_1,L_2)+(0,2), \\ \ldots & & \end{array}$$

Figure 7 depicts the situation for any fixed (ℓ_1, ℓ_2). All shifts (k_1, k_2) of $\phi_\Xi(\sigma x_1 - \cdot, \sigma x_2 - \cdot)$ whose home position falls in the striped area will contribute a nonzero entry to the column corresponding to (ℓ_1, ℓ_2). A bounding box is outlined in the figure for this area. As (ℓ_1, ℓ_2) progresses through the diagonal sequence, the union of such bounding boxes fills a rectangle at the lower left of the positive lattice orthant, since only positive shifts (k_1, k_2) are at issue in forming $\psi_\Xi(x_1, x_2)$ as a linear combination of shifts of $\phi_\Xi(\sigma x_1, \sigma x_2)$. This provides the insight for justifying the diagonal sequencing of rows as well.

The resulting matrix P is workable, although the columns have gaps in the nonzeros. A denser matrix has been obtained in [8] by "cheating" and using the following property of box splines:

$$\begin{aligned}
&\langle \psi_\Xi(x_1, x_2), \phi_\Xi(x_1 - \ell_1, x_2 - \ell_2)\rangle \\
&= \sum_{k_1,k_2} \sum_{r_1,r_2} q_{k_1,k_2} p_{r_1-\sigma\ell_1, r_2-\sigma\ell_2} \\
&\qquad \langle \phi_\Xi(\sigma x_1 - k_1, \sigma x_2 - k_2), \phi_\Xi(\sigma x_1 - r_1, \sigma x_2 - r_2)\rangle \\
&= \sum_{k_1,k_2} \sum_{r_1,r_2} q_{k_1,k_2} p_{r_1-\sigma\ell_1, r_2-\sigma\ell_2} \\
&\qquad \phi_{\Xi\cup\Xi}(r_1 - k_1 + \mu_1 + \mu_3, r_2 - k_2 + \mu_2 + \mu_3)
\end{aligned}$$

to predict a better row ordering. This result provides the prediction that

$$\begin{aligned}
\sigma\ell_i - \mu_i - \mu_3 + 1 \le\, & k_i \le \sigma\ell_i + \sigma(\mu_i + \mu_3) - 1 \\
1 - \mu_i + \mu_3 \le\, & \ell_i \le \mu_i + \mu_3 + \mathrm{div}\{(2(\mu_i + \mu_3 - 3)/\sigma\} \\
& i = 1, 2
\end{aligned}$$

which, in turn, outlines an area of lattice points that can be mapped more densely into the matrix P. Full details, with *Maple* programs, are given in [8].

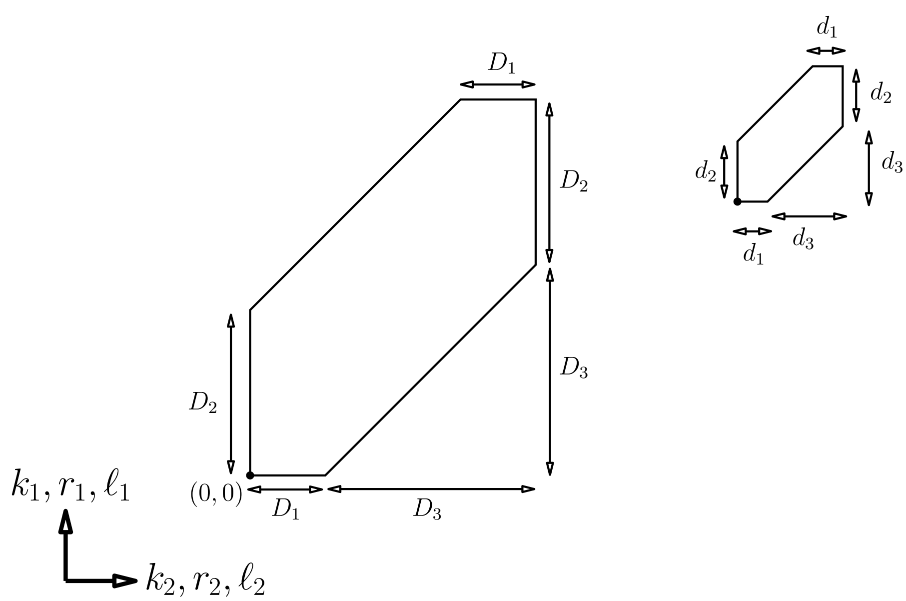

Fig. 6. Box spline supports.

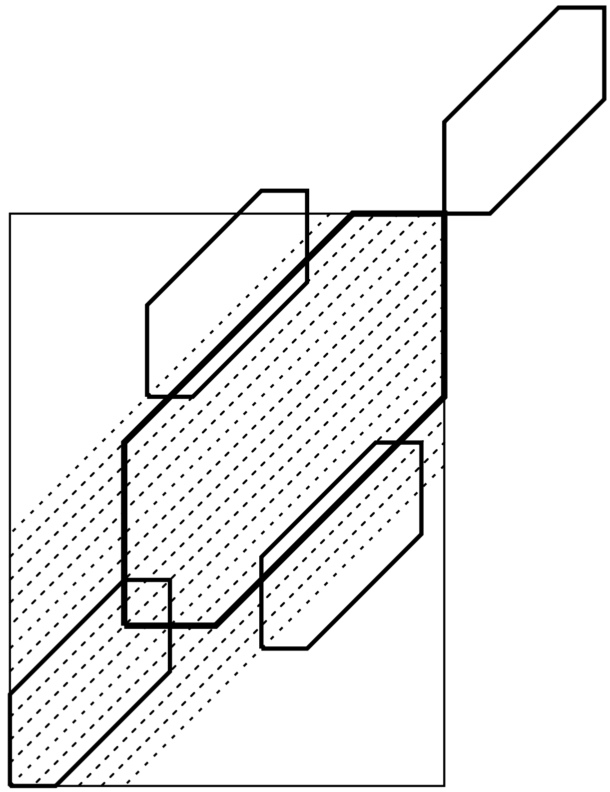

Fig. 7. Overlap of box spline supports.

The number of nonzeros in a matrix column for box splines do not generally allow one to isolate a single nullspace vector, leaving a degree of freedom to be accounted for. In the reference just cited this is handled by specifying either symmetric or antisymmetric wavelets. As an antisymmetric example: direction vectors $(1,0),(0,1),(1,1)$ with multiplicities $1,1,2$ yield the nullspace vector

$$\begin{array}{llll}
q_{0,0}=1 & q_{0,1}=3 & q_{0,2}=2 & q_{1,0}=-3 \\
q_{1,1}=-55 & q_{1,2}=-84 & q_{1,3}=-32 & q_{2,0}=2 \\
q_{2,1}=84 & q_{2,2}=353 & q_{2,3}=357 & q_{2,4}=86 \\
q_{3,1}=-32 & q_{3,2}=-357 & q_{3,3}=-791 & q_{3,4}=-552 \\
q_{3,5}=-86 & q_{4,2}=86 & q_{4,3}=552 & q_{4,4}=791 \\
q_{4,5}=357 & q_{4,6}=32 & q_{5,3}=-86 & q_{5,4}=-357 \\
q_{5,5}=-353 & q_{5,6}=-84 & q_{5,7}=-2 & q_{6,4}=32 \\
q_{6,5}=84 & q_{6,6}=55 & q_{6,7}=3 & q_{7,5}=-2 \\
q_{7,6}=-3 & q_{7,7}=-1 & &
\end{array}$$

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