Cubic precision Clough-Tocher interpolation

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The standard Clough-Tocher split-domain scheme constructs a surface element with quadratic precision. In this paper, I will look at methods for improving the degrees of freedom in Clough-Tocher schemes. In particular, I will discuss modifications to the cross-boundary construction that improve the interpolant from quadratic precision to cubic precision.

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In the general scattered data interpolation problem, we are trying to find a smooth (at least C^1 continuous) bi-variate function $F(x, y)$ such that F interpolates a set of data values at prescribed locations, i.e., $F(x_i, y_i) = z_i$ for $i = 1, \ldots, N$. In triangular scattered data fitting, we also have a set of triangles $\mathcal{T} = \{T_0, \ldots, T_{n-1}\}$ that form a proper triangulation [9] with the vertices of $T \in \mathcal{T}$ being from $\{(x_i, y_i)\}$. Commonly, we will also have normals (first partial derivatives) at the data points.

We could try to fit a single cubic patch per triangle, which we will express in Bézier form as in Figure 1 (see Farin's book [5] for details on triangular Bézier patches). For the patch to interpolate the data points and normals, the V_i and the T_{ij} are uniquely determined. This leaves us a single center control point. Unfortunately, the single degree of freedom in this control point is inadequate

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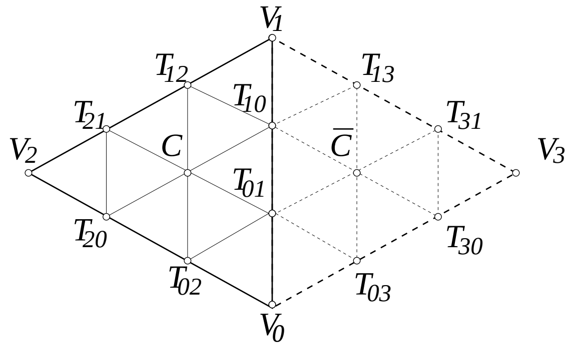

Fig. 1. Control points of a cubic. The dashed line segments show a neighboring patch.

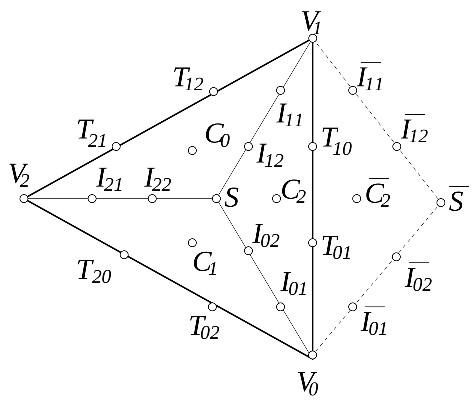

Fig. 2. Clough-Tocher control points. The dashed line segments and the "barred" points are on the mini-triangle of the neighboring macro-triangle.

to achieve C^1 continuity across all three boundaries of the patch. If only a C^0 interpolant is required, then one setting for this center point is the *quadratic precision* point of Farin [3].

One way to build a piecewise cubic, C^1 function is to split each data triangle (henceforth refered to as a *macro-triangle*) at its centroid, and fit three patches to each of the subtriangles (henceforth referred to as *mini-triangles*) (Figure 2). This was the approach taken by Clough-Tocher [1].

Using a Clough-Tocher split, we again see that the corner points of the macro-triangle (the V_i) are uniquely determined by the data points, and the T_{ij} are uniquely determined by the data normals. We now have one C_i for each macro-triangle boundary, which we can use to achieve C^1 continuity with mini-triangle within the corresponding neighoring macro-triangle. The value of the remaining control points are uniquely determined by the C^1 continuity conditions across the mini-triangle boundaries.

It is important to note that in this contruction there are only three degrees of freedom per set of mini-triangles. The values of the V_i, T_{ij}, and I_{i1} have a unique setting for interpolating the data points and data normals, and once the C_i are set, the I_{i2}, and S have unique settings for achieving C^1 continuity across the mini-triangle boundaries. The only degrees of freedom are in the settings of the C_i, each of which has a single linear degree of freedom.

Clough-Tocher used a simple linearly varying crossboudary derivative to set the degree of freedom in the C_i, giving an interpolant with quadratic precision [8]. However, more principled uses of these degrees of freedom can result in improved shape, as shown by several researchers. Kashyap [2] provides a good survey of Clough-Tocher interpolants, discussing the following methods:

- The C^0 quadratic precision patch that fits a single cubic to each macro-triangle;
- The original C^1 Clough-Tocher interpolant;
- The Farin-Kashyap [6] C^0 interpolant that has cubic precision;
- The Farin [4] C^1 interpolant that attempts to minimize the C^2 discontinuity across macro-triangle boundaries;
- A new C^1 scheme for minimizing the C^2 discontinuities across mini-triangle boundaries; note that this scheme reproduces a subspace of cubic polynomials, but does not reproduce all cubic polynomials;
- An iterative scheme that repeatedly minimizes the C^2 continuity across macro- and mini-triangle boundaries, using the previously constructed surface as a starting point at each step.

These schemes are all attempting to achieve several goals: C^1 continuity, minimization of C^2 discontinuity, and cubic precision. However, none of the above schemes has both C^1 continuity and cubic precision. In the rest of this paper, I will present methods for achieving these two goals.

To understand the new method, we will first look more closely at Farin's approach [4]. Farin's approach is to initially fit a single cubic to the macro-triangle. The center point of the patch is constructed to obtain quadratic precision [3]. This quadratic precision patch will meet its neighbors with only C^0 continuity. To achieve C^1 continuity, Farin subdivides this patch to get initial settings of all the control points, and then adjusts the center point of each mini-triangle to minimize the C^2 discontinuity across the corresponding macro-boundary. After computing all three center points, Farin then continues the Clough-Tocher construction to reestablish C^1 continuity across mini-triangle boundaries. Because Farin starts with a quadratic precision patch, this method has only quadratic precision.

The key to getting cubic precision is to realize that Farin minimizes the C^2 discontinuity between two *mini-triangles* of adjacent macro-triangles. To get cubic precision, we take a different approach: For each edge of a macro-triangle T, consider the two macro-triangles adjacent to this edge. Use Farin's method to minimize the C^2 discontinuity between cubic patches fit to these two *macro-triangles*. Next, subdivide this macro-triangle, keeping the C_i point adjacent to the edge over which we have just minimized. Repeat this for the other two edges of T. The points I_{i1} are positioned to interpolate the normal data of T. And the remaining points (I_{i2} and S) are uniquely determined by the C^1 conditions across the mini-triangle boundaries.

This scheme will both be C^1 and has cubic precision. The continuity follows from the Clough-Tocher constuction. To see cubic precision, note that if the data at two adjacent macro-triangles comes from a single cubic function, then Farin's C^2 minimization scheme will reproduce this cubic (as it will have no C^2 discontinuity). If the data at T and all three adjacent macro-triangles comes from the same cubic, then the application of Farin's method to each boundary of T will produce the same cubic. Since the setting of the remaining interior points to achieve C^1 continuity is unique, this construction will reproduce the original cubic over T.

The use of Farin's C^2 minimization method is not the only way to achieve cubic precision. All that is required is a crossboundary construction with cubic precision. For example, instead of using Farin's C^2 minimization construction, we can instead use the crossboudary of Foley-Opitz [7], who set C of the macropatch to reproduce a cubic and be C^1 otherwise.

The equations for these construction can be found in the original papers, and a full set of equations for these new C^1 cubic precision Clough-Tocher interpolant can be found in an expanded version of this paper [8].

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