Polynomial Precision Clough-Tocher Interpolation

Stephen Mann

Abstract. In this paper, I discuss a method for fitting a piecewise polynomial surface to triangulated data by first building a C^0, polynomial precision patch network to interpolate the data, then performing a 3:1 Clough-Tocher split, and finally adjusting control points to obtain higher order continuity between the patches. The continuity adjustments leave the control points unchanged if the patches already meet with the desired continuity, which gives us a Clough-Tocher scheme that has polynomial precision.

§1. Introduction

There are many solutions to the following problem: Given a triangulated set of points in the plane, with height values and derivatives at the points, find a smooth function that interpolates the values and derivatives at the data points. Here smooth means for the surface to be C^k, with $k \geq 0$. See one of several survey papers for an overview of schemes to solve this problem [5,6]. In this paper, I will investigate piecewise polynomial schemes for interpolating such data. One standard method for interpolating this data with polynomials is to split the data triangles into three or more pieces [2,13], which allows for a C^1 construction with compatible twist vectors at the corners of the patches.

I am interested in constructions that achieve polynomial precision.

Definintion 1. *A data fitting scheme S is said to have* polynomial precision *if, when given data that comes from and completely determines a degree n polynomial P, reconstructs P from this data.*

Similar constructions exists for (and this work is based on) split domain schemes [1,8] and for non-polynomial schemes [4]. The schemes I present here perform a Clough-Tocher split, fitting three polynomial patches to each data triangle.

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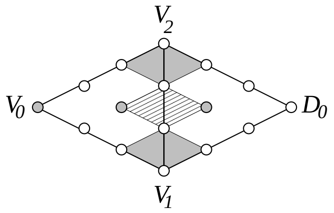

Fig. 1. Two cubics meeting along a common boundary.

Through out I will assume that we have a non-degenerate triangulation of the data with no overlapping triangles. I will use the multi-variate Bernstein-Bézier representation for polynomials with barycentric coordinates; for details, see any introductory text on CAGD, such as Farin's [3]. Many of the figures in this paper will be of schematic form; although the control points are placed regularly in a plane in the diagrams, they represent points in three-space. Further, for a polynomial F, I will use ϖF to denote the blossom of F; see Ramshaw for details on blossoming [14].

The approach in this paper is to use the derivative information to obtain polynomial precision with as low of degree patch as possible. Once the degree is determined, the maximum continuity with which the patches meet is also determined.

§2. Continuity

The continuity conditions for functional triangle Bézier patches are easily described in terms of the control points. Here I will just describe the C^0 and C^1 conditions; higher order conditions can be found elsewhere [9], with a bit of discussion of C^2 conditions appearing below.

For two patches of the same degree to meet with C^0 continuity, they must have identical control points along their shared boundary. For two cubic patches to meet with C^1 continuity, adjacent panels of the control nets must be coplanar. For example, as illustrated schematically in Figure 1, two cubic patches must share their four boundary control points. To meet with C^1 continuity, each of the three panels of four control points must be coplanar.

The observation of Foley and Opitz [4] is that if the two center panels are not coplanar, we may make them coplanar by extending each panel over the other and averaging (Figure 2). Mann later generalized this averaging scheme to higher order continuity. For example, to have C^2 continuity, groups of nine control points along the shared boundary between two patches have to be in a special configuration as illustrated in Figure 3. In the left diagram, we see the nine control points, with the three white control points being from the shared boundary. The patches are assumed to already meet with C^1 continuity, as indicated by the coplanar dark gray panels. If the two patches meet with C^2 continuity, then extending the

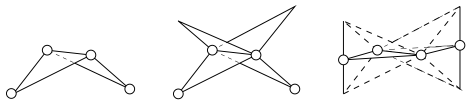

Fig. 2. Adjusting panels to be coplanar..

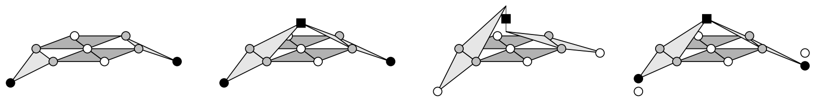

Fig. 3. Far Left: C^2 control points. Mid-Left C^2 condition. Mid-Right, Right: C^2 adjustment.

two light gray panels will yield the same point as illustrated in the second diagram of Figure 3. If instead, we already have C^1 continuity, but do not have C^2 continuity, we are in the situation illustrated in the third diagram of Figure 3. If we average the two points of the extended light gray panels, and extend these panels backwards, we obtain the situation on the right of Figure 3, which is a valid C^2 configuration. Mann gave a generalization of these C^1 and C^2 adjustments schemes to obtain C^k continuity, and showed that if both patches have degree p polynomial precision before averaging, then they will continue to have degree p polynomial precision after averaging [9,11].

Such an averaging scheme is sufficient to obtain polynomial precision interpolants using a single polynomial patch per triangle [9]. However, for C^k continuity, the patches will be of degree $4k+1$. In this paper, I will show how to use such adjustment schemes with Clough-Tocher interpolants and use lower degree patches for C^1 and C^2 continuity. An additional continuity averaging scheme (described in the next section) will be needed for the C^2 Clough-Tocher scheme.

§3. Alternative C^2 Adjustment

The C^2 condition imposes constraints on groups of nine control points along the common boundary between two patches, in particular

$$\varpi F(a^i, b^j, c, d) = \varpi G(a^i, b^j, c, d)$$

must hold for all $i+j = n-2$, where F is a polynomial over domain $\triangle abc$ and G is a polynomial over domain triangle $\triangle bad$. In some of the constructions below, we will have a configuration of nine control points where seven of the points are already set, and we want to set the remaining two control points so that the group satisfies the C^2 condition. For simplicity in the equations below I will drop the a^i, b^j arguments to the blossom.

Consider Figure 4, where we have the control points and blossom values of a quadratic defined over two domain triangles $\triangle abc$ and $\triangle bad$.

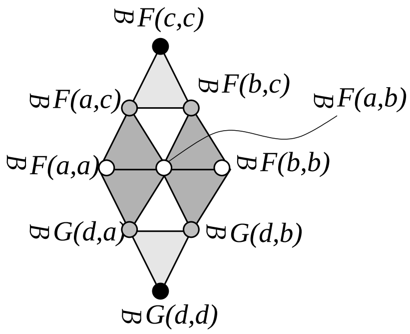

Fig. 4. Alternative C^2 adjustment scheme.

We have already seen how to adjust $\varpi F(c,c)$ and $\varpi G(d,d)$ to achieve C^2 continuity without losing polynomial precision when the nine points are already in a C^1 configuration. In this section, I show how to set $\varpi F(b,c)$ and $\varpi G(d,b)$ to get C^2 continuity when the nine points are in a C^1 configuration.

Using the blossom of the polynomial, if we have C^1 continuity and we want to adjust $\varpi F(b,c)$ and $\varpi G(d,b)$ to get C^2 continuity, then the following two equations must hold:

$$\varpi F(b,c) = \alpha\varpi F(b,b) + \beta\varpi F(a,b) + \gamma\varpi G(d,b),$$

$$\alpha\varpi G(d,b)+\beta\varpi G(d,a)+\gamma\varpi G(d,d) = \alpha'\varpi F(b,c)+\beta'\varpi F(c,a)+\gamma'\varpi F(c,c),$$

where $c = \alpha b + \beta a + \gamma d$ and $d = \alpha' b + \beta' a + \gamma' c$. Note that $\alpha' = -\alpha/\gamma$.

Substituting the first equation into the second and solving for $\varpi G(d,b)$ gives us

$$\begin{aligned}(\alpha - \alpha'\gamma)\varpi G(d,b) &= 2\alpha\,\varpi G(d,b)\\ &= \alpha'\Big(\alpha\varpi F(b,b) + \beta\varpi F(a,b)\Big)\\ &\quad - \beta\varpi G(d,a) - \gamma\varpi G(d,d) + \beta'\varpi F(c,a) + \gamma'\varpi F(c,c).\end{aligned}$$

Thus, for a non-degenerate triangulation, we have a unique solution. Note that uniqueness implies that if the nine control points come from a single polynomial (of degree two or higher), then this method for setting $\varpi F(b,c)$ and $\varpi G(d,b)$ will leave them unchanged and polynomial precision is retained.

§4. A C^0 Polynomial Precision Interpolant

As a first step for my Clough-Tocher interpolants, I will fit a single polynomial patch to the data at the vertices of a triangle. For this work, we will assume that we have $\lfloor 2n/3 \rfloor$ full derivatives at each data point. We will then set the control points of a patch closest to each data point to cause the patch to interpolate the derivatives at the data point. For control points equidistant to two data points, we use the average of the settings given by the derivatives at each data point; for patches of degree $n = 3r$, we set the center point to be the average of the three settings given by the derivatives at all three data points.

If we use this single-patch-per-face construction on a set of triangulated data, the result will be a C^0 surface that interpolates the positions and many of the derivatives at the data points. Further, if the data at the corners of a single triangle comes from a single polynomial P, then the patch fit to that triangle will reproduce P. This polynomial reproduction property follows almost immediately from the bi-variate Bernstein polynomials forming a basis, since by construction the patch will interpolate $(n+1)(n+2)/2$ pieces of data (position and derivatives) of this polynomial, which is the size of the space of degree n bi-variate polynomials.

§5. Clough-Tocher Interpolants

When fitting a single polynomial patch per triangle, we either have to construct a patch of high degree relative to the continuity (as was done in the previous section) or we need to solve the vertex consistency problem [12]. An alternative to these two choices is to fit more than one patch per face, as done by Clough-Tocher [2] and Powell-Sabin [13]. In this section, I give the general idea of using my continuity adjustment ideas to build polynomial precision Clough-Tocher interpolants, which fits three polynomial patches per triangle.

The basic process of my method is as follows:

Algorithm 2.

1) *Fit a single polynomial patch to the data as described in the previous section.*
2) *Perform 3:1 subdivision on the patch.*
3) *Adjust the control points to increase the continuity with which the patches meet (using the methods described in sections 2 and 3 for C^1 and C^2 continuity).*

This leads to the following theorem:

Theorem 3. *The piecewise polynomial surface created with Algorithm 2 has polynomial precision.*

Proof: The first step creates a patch with polynomial precision. This polynomial precision is unaffected by the 3:1 subdivion of the second step. The third step uses the continuity adjustment schemes described earlier

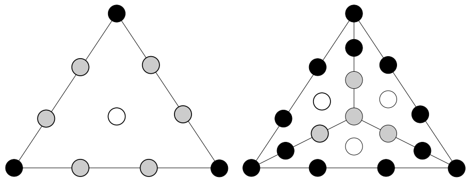

Fig. 5. Cubic Clough-Tocher control points.

in this paper, all of which preserve polynomial precision. Thus, the interpolation methods described in this paper have polynomial precision. □

The next three sections describe the process in more detail for degrees 3 and 8, with a brief discussion of degrees 4–7.

§6. Cubic Precision Clough-Tocher

Given a data triangle $\triangle P_0P_1P_2$ with first and second derivatives at the P_i, I first fit a single cubic patch to the data as follows: Using the control point shading from Figure 5 left, the black points are set using the positions at the P_i; the gray points are set using the first derivatives at the P_i; and the white point is set by averaging the settings given by the mixed partials at the three P_i. I then perform a 3:1 split of this patch, and (using the control point shading from Figure 5, right), I set the z-values of the control points with the following steps:

1) The black control points come from the subdivided patch;
2) The white control points are set to get a C^1 join across the *macro-triangle* boundaries (the boundaries of the data triangle) using the C^1 adjustment scheme described earlier in sect 2.
3) The gray control points are set to get a C^1 join across the *mini-triangle* boundaries (the boundaries between the subtriangles). This is the standard averaging step of Clough-Tocher schemes, with each edge gray point being the average of the nearest edge black point and the two adjacent white points, and with the center gray point being the average of the three edge gray points.

The resulting patches meet with C^1 continuity across both macro- and mini-boundaries, and has cubic precision as discussed earlier. Further, the three patches meeting at the split point meet with C^2 continuity at this point [15]. This is seen in Step 3 of the construction, used to set the gray points, which is the de Casteljau evaluation of a quadratic patch (starting with the black edge points and the white points). Such a construction is said to have *super-continuity* at the split point, and results in good stability properties [7].

Note that the cubic precision Clough-Tocher construction described in this section is similar to one I published earlier [8], but differs in several

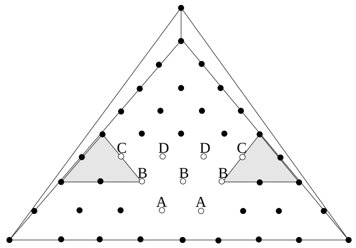

Fig. 6. Octic Clough-Tocher control points.

details. In particular, the earlier scheme only required positions and first derivatives at the data points, but used data from neighboring triangles to obtain cubic precision.

§7. Degrees Four Through Seven

Using a quartic Clough-Tocher interpolant gives no improvement in continuity over the cubic Clough-Tocher interpolant. Using a quintic Clough-Tocher interpolant gives enough flexibility to get C^2 continuity over the macro-boundaries, but only C^1 continuity over the mini-triangle boundaries. Likewise, the sextic and Septic Clough-Tocher interpolants have C^2 continuity across the macro-boundaries, but only C^1 continuity across the mini-boundaries. However, despite no increase in continuity, these schemes have top order polynomial precision. Details can be found in two technical reports [9,10].

§8. Octic Precision Clough-Tocher

The seventh degree Clough-Tocher interpolant almost has enough degrees of freedom to obtain C^2 continuity on the mini-boundaries. However, the last step of the construction has an unsolvable extra cycle of constraints, leaving us with a C^1 scheme. When we increase the degree to eight, this cycle is broken and we get a Clough-Tocher scheme with C^2 continuity across both macro- and mini-triangle boundaries. This octic scheme proceeds as follows:

1) Fit a single octic patch to interpolate position, first, second, third, and fourth derivatives (although the fourth derivatives are interpolated only if adjacent vertices come from a common octic surface) as described in Section 4.
2) Perform a 3-to-1 split on each patch.
3) Adjust the interior control points using the labeling in Figure 6 as follows (in this figure, the split point should be at the centroid of the outer triangle, but the triangle shapes have been distorted to better see the control points of a single patch; in the following adjustments, the A, B, and C points are adjusted in all three mini-patches):

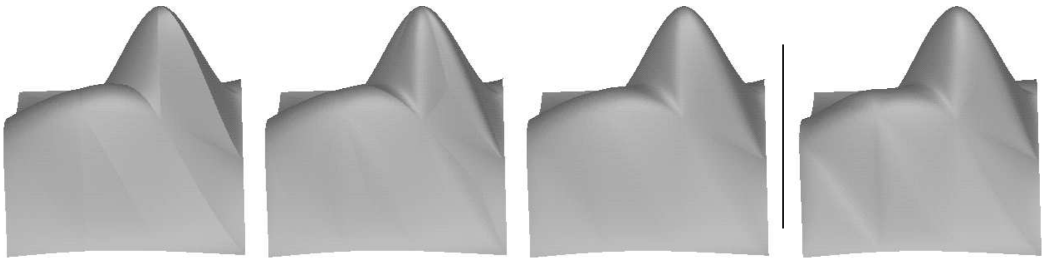

Fig. 7. Process for cubic precision Clough-Tocher.

A) Set using the Foley-Opitz C^1 construction (Section 2).

B) Set using the C^2 adjustment across macro boundary (Section 2).

C) Set using the alternative C^2 adjustment across mini-triangle boundary (Section 3).

D) Set using the C^2 adjustment across mini boundary (Section 2).

The result is a construction for which we need position, first, second, third, and fourth derivatives at the data points, with the resulting surface have the following properties:

1) Octic precision if the data at the triangle and the three adjacent triangles comes from a common octic.
2) C^2 across all boundaries.
3) C^4 at split point.

The C^4 continuity at the split point results from setting the control of the three patches (near the split point) as the control points of the subdivided patch.

§9. Results

In Figure 7 we see an illustration of the construction process described in Section 6. On the left is the C^0 surface consisting of a single cubic per data triangle. The C^0 discontinuities between adjacent patches are clearly visible. In the second from the left is the surface resulting from just adjusting the split patches to have C^1 continuity across macro-boundaries. And second from the right is the final Clough-Tocher surface, having C^1 continuity across all boundaries. For comparison, on the right is the surface constructed by the original Clough-Tocher interpolant, which only has quadratic precision.

In Figure 8, we see the shaded images and curvature plots of octic surfaces fit to the same sampling of the Frankye function. In the left column is the C^1 surface consisting of a single octic patch per data triangle (i.e., the surface constructed after Step 1 of section 8, followed by an application of the C^1 continuity adjustment scheme). On the right is the C^2 octic precision Clough-Tocher surface described in Section 8. Although

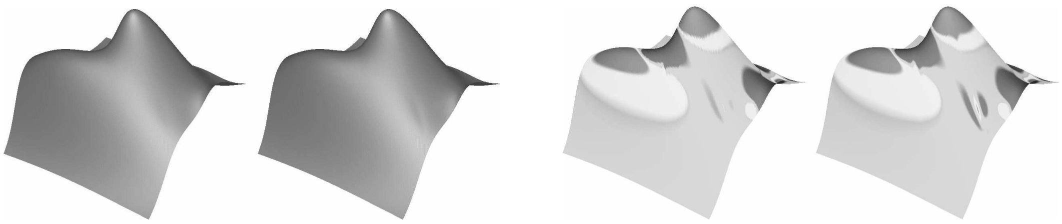

Fig. 8. Octic surface and curvature plots.

the latter surface has high continuity between patches (as can be seen in the curvature plots), the interior of some of these patches have worse shape, as can be seen by the divot on the right side of the surface.

What we see from these results is that (a) the cubic polynomial precision Clough-Tocher scheme gives significant visual improvements over the single cubic patch per triangle scheme. This improvement results primarily from getting rid of the C^1 discontinuities. While my cubic method also gives a significant improvement over the standard Clough-Tocher technique, problems with the standard Clough-Tocher method are well known and many improved cubic versions exist.

However, the C^2 octic polynomial precision Clough-Tocher scheme described here appears to give worse results than the C^1 single octic patch per triangle scheme. This degradation in shape likely results from the adjustment scheme, which while it retains polynomial precision, it undoubtedly increases the constant in the error term. Whether this loss in shape is worth the higher continuity will depend on the particular application using the scheme. Note also that for the examples used in this paper, the samplings of Franke function were sparse. With denser samplings, many of the problems with the C^2, octic Clough-Tocher scheme go away.

Finally, note that Alfeld and Schumaker were able to construct a C^2 polynomial precision Clough-Tocher element of degree 7 [1], but have to specify derivatives along the boundaries of the triangles. In my approach, I specify no boundary derivatives, but I had to use degree 8 patches to achieve C^2 continuity; my attempt using degree 7 patches did not have enough degrees of freedom to get C^2 continuity [9]. Potentially a modification of my approach could be used to get a C^2 construction of degree 7 without requiring boundary derivatives.

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Stephen Mann
School of Computer Science
University of Waterloo
200 University Ave W
Waterloo, Ontario, N2L 3G1, CANADA
`smann@uwaterloo.ca`