Adjusting Control Points to Achieve Continuity

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Abstract

In this paper, I discuss a method for increasing the continuity between two functional triangular polynomial patches by adjusting their control points. The method described in this paper leaves the control points unchanged if the patches already meet with the desired level of continuity. As an example of using this construction, I give a simple method to construct a C^0 patch network with high order polynomial precision, and then adjust its control points to increase continuity without decreasing the order of polynomial precision.

Key words: Continuity, triangular Bézier patches, polynomial precision.

1 Introduction and Background

Surface modeling commonly uses piecewise polynomial surfaces. Adjacent patches are constructed to join with some order of continuity, with the intent of hiding the seams between patches. For surfaces having a rectangular topology, we can use tensor product B-splines, giving us a patch network with patches that automatically meet with maximal continuity. However, tensor product surfaces are not well suited to some data sets, and often triangular schemes are preferable.

While there is a triangular form of B-splines [1] for functional data, the evaluation algorithm is complex and computationally expensive. So for triangular data, a patch network is commonly constructed by explicitly setting control points to achieve continuity. The continuity problem along with the vertex consistency problem are the driving forces in these constructions. The nature

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of the problem results in a fairly high degree patch being required to solve the continuity/vertex consistency problem, and results in a large number of degrees of freedom left unspecified in the patch once these twin problems are solved. These degrees of freedom are usually set ad hoc or left as shape parameters without any suggestions for default settings. Unfortunately, these shape parameters are critical to surface quality, and poor settings of them result in surfaces with poor shape.

A few researchers have taken a different approach. In particular, Foley and Optiz [4] construct a C^1 rational patch network that interpolates triangulated data having position and first derivatives at the data points. One of the key ideas of their construction is to first construct a patch network that has cubic precision and then adjust the patches control points to obtain C^1 continuity while maintaining cubic precision. Their idea of first constructing patches that have cubic precision and then adjust the control points to obtain C^1 continuity was later used to derive a cubic precision Clough-Tocher scheme [6].

In this paper, I will review the Foley-Opitz C^1 adjustment, and then show how it can be generalized to adjust the control points of triangular patches to achieve any level of continuity. These adjustments leave the control points unchanged if the patches already meet with the desired level of continuity, with the result that if the patches have a particular polynomial precision before the adjustment, then they retain that level of polynomial precision after the adjustment. I will then give a simple example to illustrate how to use the continuity adjustment scheme.

1.1 Background

I will use the multi-variate Bernstein-Bézier representation for scalar valued polynomials. The description of the Bernstein-Bézier representation that I give here just touches on the topics that I need for this paper. For a more complete discussion on triangular Bézier patches, see [2] or any introductory text on CAGD.

I will index the control points using standard multiindex notation. Figure 1 is a schematic illustrating this labeling for quintic patches. Many of the figures in this paper will be of this schematic form; although the control points in the diagram are in a plane, they represent points in three-space. However, labels V_0, V_1, V_2, and D_0 will always refer to corner points of the domain triangles in these figures. And while this paper considers functional Bézier patches, I will use the barycentric form of the polynomials, where every point in a domain triangle is expressed as an affine combination of the triangle corners, $\triangle V_0V_1V_2$:

$$t = t_0V_0 + t_1V_1 + t_2V_2,$$

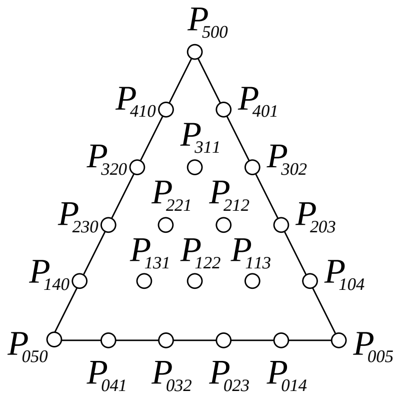

Fig. 1. Quintic Bézier polynomial with vertices labeled.

with $t_0 + t_1 + t_2 = 1$. In this formulation, triangular Bézier patches have the following form:

$$B(t) = \sum_{\vec{i}, |\vec{i}|=n} P_{\vec{i}} B^n_{\vec{i}}(t),$$

where $\vec{i} = (i_0, i_1, i_2)$ is a multiindex, the $P_{\vec{i}}$ are the coefficients (control points) for the patch, and the $B^n_{\vec{i}}$ are the degree n Bernstein polynomials:

$$B^n_{\vec{i}}(t) = \frac{n!}{i_0! i_1! i_2!} t_0^{i_0} t_1^{i_1} t_2^{i_2}.$$

The derivative and continuity analysis used in this paper is simplified by using the *polar form* or *blossom* of the polynomial [9]. For a degree n polynomial B, the polar form of B (denoted ϖB) is an n-variate function satisfying the following:

- ϖB is symmetric;
- ϖB is multi-affine in each argument;
- $\varpi B(u^{<n>}) = B(u)$,

where $\varpi B(u^{<n>})$ is ϖB evaluated with all n of its arguments equal to u. The polar form has a nice relation to the Bézier control points of a triangular patch. In particular, over a domain triangle $\triangle V_0 V_1 V_2$, $P_{i,j,k} = \varpi B(V_0^{<i>}, V_1^{<j>}, V_2^{<k>})$.

The coefficients of a Bézier patch give us information about the derivatives of the patch. In particular, the derivatives in the direction of the triangle edges are proportional to simple differences of the control points:

$$\begin{aligned} B(V_0) &= P_{n00}, \\ d_{e01} B(V_0) &= n(P_{n-1,1,0} - P_{n,0,0})/|V_1 - V_0|, \\ d_{e02} B(V_0) &= n(P_{n-1,0,1} - P_{n,0,0})/|V_2 - V_0|. \end{aligned}$$

Here, e_{01} is the unit directional derivative from V_0 to V_1; e_{02} is similar. Derivatives at the other corners and higher order derivatives are computed in a similar fashion. Thus, if we are given position and derivative information at the corners of the patch, it is easy to find settings of the control points to interpolate this information. In the following discussion, I will merely state that we set control points to match the derivatives and not give the formulas.

In my diagrams, when a group of control points are set using the derivative information at one of the V_i, I will circle those points with a dashed circle, as in Figures 4 and 13. Conversely, these dashed regions also indicate the number of derivatives needed at the corresponding V_i (i.e., at each data point, we need to have the position and the appropriate derivatives for setting the circled control points). Control points covered by more than one dashed circle will be set by averaging the values computed for each set of derivatives.

1.2 Continuity

If we want two neighboring patches to meet with C^k continuity, then there are simple settings of the control points to achieve this. To achieve a C^0 join, the boundary control points of two patches have to be identical. To achieve C^1 continuity between patches F and G over $\triangle V_0V_1V_2$ and $\triangle V_2V_1D_0$ respectively, blossoming tells us that

$$\begin{aligned}\varpi G(D_0, V_1^{<i>}, V_2^{<j>}) &= \varpi F(D_0, V_1^{<i>}, V_2^{<j>}) \\ &= d_0\varpi F(V_0, V_1^{<i>}, V_2^{<j>}) + d_1\varpi F(V_1, V_1^{<i>}, V_2^{<j>}) + \\ &\quad d_2\varpi F(V_2, V_1^{<i>}, V_2^{<j>}),\end{aligned}$$

where (d_0, d_1, d_2) are the barycentric coordinates of D_0 relative to $\triangle V_0V_1V_2$ and $i+j = n-1$, $i, j \geq 0$. Geometrically, the condition is that the four control points in each of the the neighboring triangular panels of the two patches must be coplanar (for example, in Figure 5 each shaded and hashed group of four points must be coplanar).

Lai [5] gave a general geometric construction for C^k continuity to determine if two patches that meet with C^{k-1} continuity also meet with C^k continuity. Rephrased as a blossoming condition, we have the following. For even k, the two patches meet with C^k continuity if and only if for all i, $0 \leq i \leq n-k$, $j = n-k-i$

$$\varpi G(V_1^{<i>}, V_2^{<j>}, D_0^{<k>}, V_0^{<k>}) = \varpi F(V_1^{<i>}, V_2^{<j>}, D_0^{<k>}, V_0^{<k>}), \quad (1)$$

and for odd k, the two patches meet with C^k continuity if and only if for all

For $m = 1$ to $\lfloor k/2 \rfloor$

For $i = 0$ to $n-k$ with $j = n-k-i$

$$
\begin{aligned}
&\varpi G(V_1^{<i>}, V_2^{<j>}, D_0^{<k-m>}, V_0^{<m-1>}, V_0) = \\
&\qquad v_0 \varpi G(V_1^{<i>}, V_2^{<j>}, D_0^{<k-m>}, V_0^{<m-1>}, D_0) + \\
&\qquad v_1 \varpi G(V_1^{<i>}, V_2^{<j>}, D_0^{<k-m>}, V_0^{<m-1>}, V_1) + \\
&\qquad v_2 \varpi G(V_1^{<i>}, V_2^{<j>} D_0^{<k-m>}, V_0^{<m-1>}, V_2), \\
&\varpi F(V_1^{<i>}, V_2^{<j>}, V_0^{<k-m>}, D_0^{<m-1>}, D_0) = \\
&\qquad d_0 \varpi F(V_1^{<i>}, V_2^{<j>}, V_0^{<k-m>}, D_0^{<m-1>}, V_0) + \\
&\qquad d_1 \varpi F(V_1^{<i>}, V_2^{<j>}, V_0^{<k-m>}, D_0^{<m-1>}, V_1) + \\
&\qquad d_2 \varpi F(V_1^{<i>}, V_2^{<j>} V_0^{<k-m>}, D_0^{<m-1>}, V_2),
\end{aligned}
$$

Fig. 2. Computation required by Lai's C^k continuity test.

i, $0 \le i \le n-k$, $j = n-k-i$

$$
\begin{aligned}
\varpi G(V_1^{<i>}, V_2^{<j>}, D_0^{<k>}, V_0^{<k-1>}) &= \varpi F(V_1^{<i>}, V_2^{<j>}, D_0^{<k>}, V_0^{<k-1>}) \\
&= d_0 \varpi F(V_1^{<i>}, V_2^{<j>}, D_0^{<k-1>}, V_0^{<k>}) + \\
&\quad d_1 \varpi F(V_1^{<i+1>}, V_2^{<j>}, D_0^{<k-1>}, V_0^{<k-1>}) + \\
&\quad d_2 \varpi F(V_1^{<i>}, V_2^{<j+1>}, D_0^{<k-1>}, V_0^{<k-1>}),
\end{aligned}
\tag{2}
$$

Figure 2 shows how to compute these blossom values, assuming the patches are known to meet with C^{k-1} continuity, and the blossoms values on the right hand side of the equations were computed in testing for C^{k-1} continuity, where (v_0, v_1, v_2) are the barycentric coordinates of V_0 relative to domain triangle $\triangle D_0 V_1 V_2$ (note that $v_0 = 1/d_0$, $v_1 = -d_1/d_0$, $v_2 = -d_2/d_0$).

Lai gives the geometric interpretation for these conditions; Figures 8 (left) and 10 show the C^2 and C^3 conditions respectively for triangles.

A similar construction is valid for curves, which I will look at briefly as a means of illustrating the outer loop in the above computation. For curves, you have the same computations as above, but without the inner loop, and only one of V_1 and V_2 appears as a parameter to the blossom. Figure 3 shows the construction of the blossom values of F required by Lai's construction for a quintic curve parameterized over $[0, 1]$. Part (a) shows the control points and their blossom values. Here we also see the C^0 and C^1 conditions: $g(1^{<5>}) = f(1^{<5>})$ for C^0 continuity, and the condition that $\varpi G(1^{<4>}, 2^{<1>})$ lies at X to have C^1 continuity. Part (b) shows the C^2 and C^3

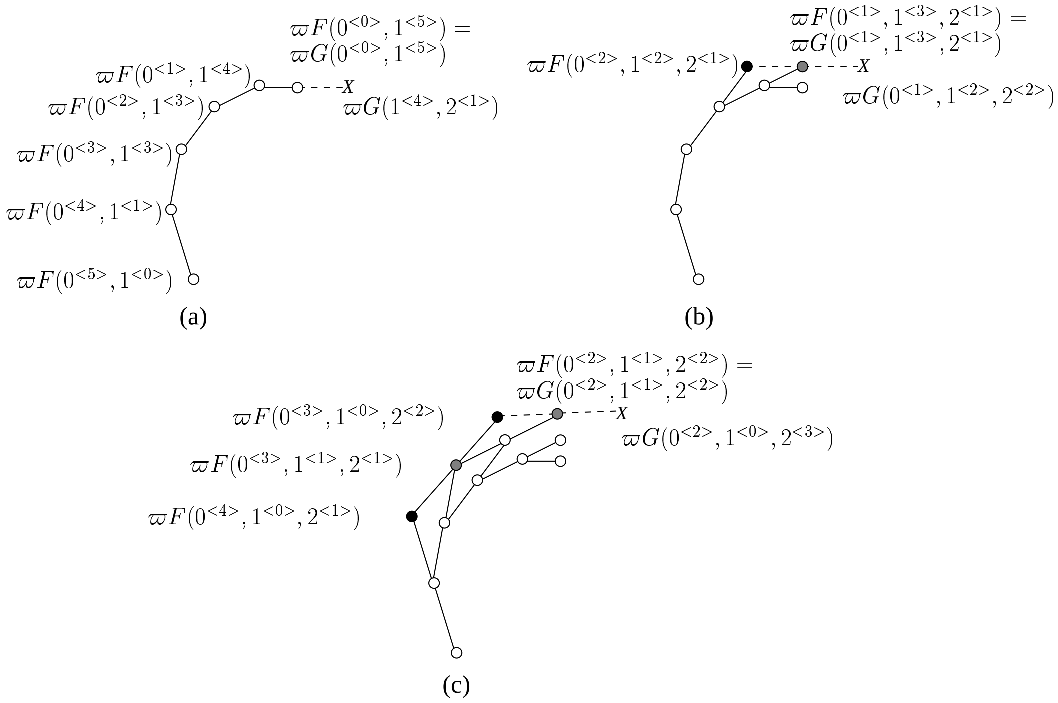

Fig. 3. Lai's continuity conditions for curves for F parameterized over $[0, 1]$ and G parameterized over $[1, 2]$.

conditions: First we construct $\varpi F(0^{<1>}, 1^{<3>}, 2^{<1>})$ (the gray point) which must be equal to $\varpi G(0^{<1>}, 1^{<3>}, 2^{<1>})$ for C^2 continuity. Then we construct $\varpi F(0^{<2>}, 1^{<2>}, 2^{<1>})$ (the black point), and require that $\varpi G(0^{<1>}, 1^{<2>}, 2^{<2>})$ lie at X for C^3 continuity. In part (c), we see the construction for C^4 continuity (the gray points) and C^5 continuity (the black points).

The construction for triangles is similar, except that there are multiple conditions for each order of continuity (as in Figures 5 and 7), and thus the inner loop in Figure 2.

The rest of this paper will proceed as follows: In the next section, I will describe an averaging technique for adjusting the control points of two neighboring patches to obtain C^k continuity between these two adjacent patches. The important feature of the averaging scheme is that if the patches already meet with C^k continuity, then the control points remain unchanged. Then in Section 3, as an example of how this continuity scheme is useful, I will present a simple method for constructing a C^0 interpolant that reproduces maximal degree polynomials. By using the averaging scheme described in this paper to increase the continuity between these patches we obtain higher order continuity without decreasing the order of polynomial precision.

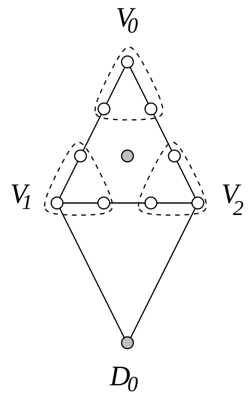

Fig. 4. Foley-Opitz cross-boundary scheme over domains $\triangle V_0V_1V_2$ and $\triangle V_2V_1D_0$.

2 Increasing Continuity Between Two Patches

Given two polynomial patches over adjacent triangles of our domain (i.e., where the two domain triangles share exactly one edge) that meet with some level of continuity, we would like to increase the continuity with which our patches meet. This increased continuity will require adjusting the control points of one or both patches. However, I will further require my adjustment scheme to leave unchanged any control points that already meet the continuity conditions. As a starting point, I use a variation of the method of Foley-Opitz [4], who devised such a construction for cubics.

Foley and Opitz were working with triangulated data having first derivatives at the data points. They were fitting *hybrid-cubics* to the data (see their paper for details on hybrid-patches), and as part of their construction they found a cubic precision construction for two adjacent triangles. This part of their method (illustrated in Figure 4) constructs a cubic patch for each data triangle. The patch for $\triangle V_0V_1V_2$ is constructed by using the data at the V_i to set the white control points, and the positional data at D_0 is used to set the shaded control point (i.e., the center point of the patch is set so that the patch interpolates the z-value at D_0 when evaluated at D_0). The patch for $\triangle V_2V_1D_0$ is built in a symmetric fashion.

As illustrated schematically in Figure 5, these two patches will share boundary control points (since both patches compute the boundary points from the data at V_1 and V_2). To meet with C^1 continuity, each of the three panels of four control points must be coplanar. The gray panels will be coplanar because both patches compute these control points consistent with the derivative information at V_1 and V_2. However, the four points on the hashed panel will not, in general, be coplanar; these four points will be coplanar if and only if the data at the V_i and at D_0 come from a common cubic.

To achieve C^1 continuity in the general case (i.e., non-coplanar hashed panels

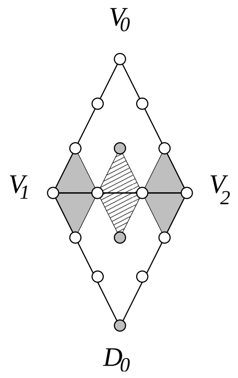

Fig. 5. Two cubics meeting along a common boundary over domains $\triangle V_0V_1V_2$ and $\triangle V_2V_1D_0$.

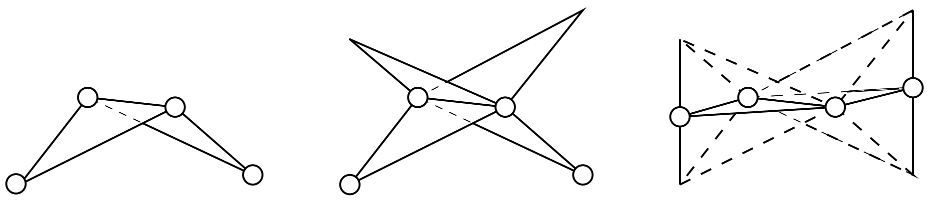

Fig. 6. Adjusting the panels to meet C^1.

panels, as illustrated on the left in Figure 6), Foley and Opitz extend both panels to the neighboring triangle as shown in the middle of Figure 6. They then average the two points on either side, which results in coplanar panels (the right in Figure 6). Note that if the data at V_1 and V_2 of Figure 4 come from a common cubic, then panels on the left of Figure 6 will be coplanar, and the averaging will have no effect. The result is that their construction builds two cubic patches that meet with C^1 continuity and reproduce cubic polynomials if the data at all four vertices comes from a common cubic.

2.1 C^0 Continuity

Most constructions build common boundary curves before setting the interior control points, so the patches meet with C^0 continuity as a first step. However, if we knew our construction might build patches that did not meet with C^0 continuity, we could use the average of each pair of control points along the boundary (one from each patch) as the boundary points. The patches would then have a common boundary, and if the initial boundary points were already identical, this averaging would not change them.

We can extend the Foley-Opitz averaging scheme to more interior vertices to achieve higher order continuity. For example, if we apply it to the next layer of control points, we can achieve C^2 continuity.

To have C^2 continuity, we first must have C^1 continuity, which we can get using the C^1 adjustment scheme. Now, given that we have C^1 continuity between patches F and G defined over $\triangle V_0V_1V_2$ and $\triangle V_2V_1D_0$ respectively, the following additional condition must hold for C^2 continuity:

$$\varpi G(V_1^{<i>}, V_2^{<j>}, D_0, V_0) = \varpi F(V_1^{<i>}, V_2^{<j>}, D_0, V_0),$$

for $i + j = n - 2$, $i, j \geq 0$. We can construct these points from the control points of the polynomials as

$$\begin{aligned}
\varpi F(V_1^{<i>}, V_2^{<j>}, D_0, V_0) = {} & d_0 \varpi F(V_1^{<i>}, V_2^{<j>}, V_0, V_0) + \\
& d_1 \varpi F(V_1^{<i>}, V_2^{<j>}, V_1, V_0) + \\
& d_2 \varpi F(V_1^{<i>}, V_2^{<j>}, V_2, V_0), \\
\varpi G(V_1^{<i>}, V_2^{<j>}, V_0, D_0) = {} & v_0 \varpi F(V_1^{<i>}, V_2^{<j>}, D_0, D_0) + \\
& v_1 \varpi F(V_1^{<i>}, V_2^{<j>}, V_1, D_0) + \\
& v_2 \varpi F(V_1^{<i>}, V_2^{<j>}, V_2, D_0),
\end{aligned}$$

where (d_0, d_1, d_2) are the barycentric coordinates of D_0 relative to $\triangle V_0V_1V_2$ and (v_2, v_1, v_0) are the barycentric coordinates of V_0 relative to the triangle $\triangle D_0V_2V_1$.

Geometrically, these conditions require certain groups of nine control points (three on the common boundary, and three on each of the two patches) to be in a special relationship as described by Farin [2] and later by Lai [5]. The vertices adjacent to the shaded triangles in each of the three diagrams in Figure 7 show the groups of vertices affecting C^2 continuity in the quartic case. The left diagram in Figure 8 illustrates these constraints. First, the dark shaded panels must be coplanar (this is the C^1 condition). Next, if we take the three vertices of F connected by the light shaded panel, and extend them in the ratio given by the two domain triangles, we get the point $\varpi F(D_0, V_0, V_1^{<i>}, V_2^{<j>})$. If we do the same extension using the six corresponding points of the neighboring patch G, we get the point $G(D_0, V_0, V_1^{<i>}, V_2^{<j>})$. For the patches to meet with C^2 continuity, these two points must be the same (the black square in the left diagram of Figure 8). Such a condition must hold at all three groups of points illustrated in Figure 7.

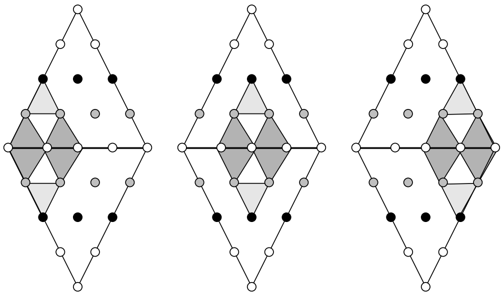

Fig. 7. Quartic control points affecting C^2 continuity.

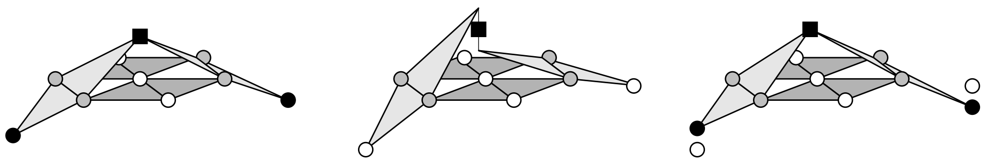

Fig. 8. The C^2 constraints (left) and construction (middle and right).

In general the groups of nine control points will not have this property, and if F and G meet with C^1 continuity along their common boundary then we have a situation more like the middle diagram of Figure 8. In this case, my scheme is to average the two extrapolated points

$$\begin{aligned}\varpi\bar{F}(D_0, V_0, V_1^{<i>}, V_2^{<j>}) &= \varpi\bar{G}(D_0, V_0, V_1^{<i>}, V_2^{<j>}) = \\ &= \Big(F(D_0, V_0, V_1^{<i>}, V_2^{<j>}) + G(D_0, V_0, V_1^{<i>}, V_2^{<j>})\Big)/2,\end{aligned}$$

giving the black square in the middle figure, and then extrapolate in the other direction to get the black points of the right diagram:

$$\begin{aligned}\varpi\bar{F}(V_0, V_0, V_1^{<i>}, V_2^{<j>}) &= v_0\varpi\bar{F}(D_0, V_0, V_1^{<i>}, V_2^{<j>}) \\ &\quad v_1\varpi F(V_1, V_0, V_1^{<i>}, V_2^{<j>}) \\ &\quad v_2\varpi F(V_2, V_0, V_1^{<i>}, V_2^{<j>}) \\ \varpi\bar{G}(D_0, D_0, V_1^{<i>}, V_2^{<j>}) &= d_0\varpi\bar{G}(D_0, V_0, V_1^{<i>}, V_2^{<j>}) \\ &\quad d_1\varpi F(V_1, D_0, V_1^{<i>}, V_2^{<j>}) \\ &\quad d_2\varpi F(V_2, D_0, V_1^{<i>}, V_2^{<j>}).\end{aligned}$$

The white points in the right diagram indicate the initial positions of the control points.

Note that if a set of nine control points is already in an acceptable C^2 configuration, then this averaging will leave the control points unchanged.

The C^1 and C^2 conditions illustrate the two types of conditions that will occur: one is a coplanarity requirement and the other a constructed point that must be common to both patches. For higher order continuity, Equations 1 and 2 give the C^k condition, assuming we already have C^{k-1} continuity. To get the relevant blossom values to test for continuity, we use Lai's construction. If Lai's continuity condition is not met, we can use the appropriate averaging scheme:

- For odd k,

$$\begin{aligned} \varpi\bar{G}(V_1^{<i>},\, V_2^{<j>}, D_0^{<k>}, V_0^{<k-1>}) = & \\ & \Big(\varpi G(V_1^{<i>}, V_2^{<j>}, D_0^{<k>}, V_0^{<k-1>})+ \\ & \quad \varpi F(V_1^{<i>}, V_2^{<j>}, D_0^{<k>}, V_0^{<k-1>})\Big)/2 \\ \varpi\bar{F}(V_1^{<i>},\, V_2^{<j>}, D_0^{<k-1>}, V_0^{<k>}) = & \\ & \Big(\varpi G(V_1^{<i>}, V_2^{<j>}, D_0^{<k-1>}, V_0^{<k>})+ \\ & \quad \varpi F(V_1^{<i>}, V_2^{<j>}, D_0^{<k-1>}, V_0^{<k>})\Big)/2 \end{aligned} \tag{3}$$

- For even k,

$$\begin{aligned} \varpi\bar{G}(\, V_1^{<i>}, V_2^{<j>}, D_0^{<k>}, V_0^{<k>}) &= \varpi\bar{F}(V_1^{<i>}, V_2^{<j>}, D_0^{<k>}, V_0^{<k>}) \\ &= \Big(\varpi G(V_1^{<i>}, V_2^{<j>}, D_0^{<k>}, V_0^{<k>}) + \varpi F(V_1^{<i>}, V_2^{<j>}, D_0^{<k>}, V_0^{<k>})\Big)/2. \end{aligned} \tag{4}$$

After averaging, we need to adjust the actual control points of the patches, which for the C^k adjustment can be done by the computation given in Figure 9 (which is essentially the reverse of the computation given in Figure 2):

This leads to the following theorem:

> **Theorem:** Given two degree n bivariate polynomials F and G in Bézier form over domains $\triangle V_0V_1V_2$ and $\triangle V_2V_1D_0$ respectively, where F and G meet with C^{k-1} continuity ($k < n$) along V_1V_2. Construct $\bar{F}$ and $\bar{G}$ to have the same Bézier control points as F and G respectively, except (for $i + j = n - k, i \geq 0, j \geq 0$) set $\varpi\bar{F}(V_0^{<k>}, V_1^{<i>}, V_2^{<j>})$ and $\varpi\bar{G}(D_0^{<k>}, V_1^{<i>}, V_2^{<j>})$ by computing $\varpi\bar{F}(V_0^{<\lfloor(k+1)/2\rfloor>}, D_0^{<\lfloor k/2\rfloor>}, V_1^{<i>}, V_2^{<j>})$ and $\varpi\bar{G}(V_0^{<\lfloor k/2\rfloor>}, D_0^{<\lfloor(k+1)/2\rfloor>}, V_1^{<i>}, V_2^{<j>})$ with Equation 3 or 4 (depending on whether k is odd or even) followed by the computation of Figure 9.
>
> Then $\bar{F}$ and $\bar{G}$ meet with C^k continuity. Further, if F and G meet with C^k continuity, then $\bar{F}$ is identical to F and $\bar{G}$ is identical to G.

For $m = \lfloor k/2 \rfloor$ downto 1

For $i = 0$ to $n-k$ with $j = n-k-i$

$$\begin{aligned}
&\varpi\bar{G}(V_1^{<i>}, V_2^{<j>}, D_0^{<k-m>}, V_0^{<m-1>}, D_0) = \\
&\quad d_0\varpi\bar{G}(V_1^{<i>}, V_2^{<j>}, D_0^{<k-m>}, V_0^{<m-1>}, V_0) + \\
&\quad d_1\varpi G(V_1^{<i>}, V_2^{<j>}, D_0^{<k-m>}, V_0^{<m-1>}, V_1) + \\
&\quad d_2\varpi G(V_1^{<i>}, V_2^{<j>} D_0^{<k-m>}, V_0^{<m-1>}, V_2), \\
&\varpi\bar{F}(V_1^{<i>}, V_2^{<j>}, V_0^{<k-m>}, D_0^{<m-1>}, V_0) = \\
&\quad v_0\varpi\bar{F}(V_1^{<i>}, V_2^{<j>}, V_0^{<k-m>}, D_0^{<m-1>}, D_0) + \\
&\quad v_1\varpi F(V_1^{<i>}, V_2^{<j>}, V_0^{<k-m>}, D_0^{<m-1>}, V_1) + \\
&\quad v_2\varpi F(V_1^{<i>}, V_2^{<j>} V_0^{<k-m>}, D_0^{<m-1>}, V_2).
\end{aligned}$$

Fig. 9. Computation to compute $\varpi\bar{F}(V_0^{<k>}, V_1^{<i>}, V_2^{<j>})$ and $\varpi\bar{G}(D_0^{<k>}, V_1^{<i>}, V_2^{<j>})$.

Proof: The averaging step (Equations 3 and 4) sets the points or panels into alignment to satisfy Lai's continuity condition (Equations 1 and 2). The backward adjustments step above sets $\varpi\bar{F}(V_0^{<k>}, V_1^{<i>}, V_2^{<j>})$ and $\varpi\bar{G}(D_0^{<k>}, V_1^{<i>}, V_2^{<j>})$ such that running the forward computation (Figure 2) for $\bar{F}$ and $\bar{G}$ results in the points we computed with Equations 3 and 4, and thus the modified patches will meet with the desired continuity.

If F and G meet with C^k continuity, then the averaging step leaves $\varpi\bar{F}(V_0^{<\lfloor (k+1)/2 \rfloor>}, D_0^{<\lfloor k/2 \rfloor>}, V_1^{<i>}, V_2^{<j>})$ and $\varpi\bar{G}(V_0^{<\lfloor k/2 \rfloor>}, D_0^{<\lfloor (k+1)/2 \rfloor>}, V_1^{<i>}, V_2^{<j>})$ unchanged, and thus computation of Figure 9 sets

$$\varpi\bar{F}(V_0^{<\lfloor (k+1)/2 \rfloor>}, D_0^{<\lfloor k/2 \rfloor>}, V_1^{<i>}, V_2^{<j>}) = \varpi F(V_0^{<\lfloor (k+1)/2 \rfloor>}, D_0^{<\lfloor k/2 \rfloor>}, V_1^{<i>}, V_2^{<j>})$$

and

$$\varpi\bar{G}(V_0^{<\lfloor k/2 \rfloor>}, D_0^{<\lfloor (k+1)/2 \rfloor>}, V_1^{<i>}, V_2^{<j>}) = \varpi G(V_0^{<\lfloor k/2 \rfloor>}, D_0^{<\lfloor (k+1)/2 \rfloor>}, V_1^{<i>}, V_2^{<j>})$$

making F and $\bar{F}$ identical, and G and $\bar{G}$ identical.

Note that the C^k adjustment does not affect the control points that influence lower order derivatives along the common boundary of F and G.

2.4 C^3 example

As an example of the averaging and adjustment process, consider the C^3 condition, which after construction of the blossom values is

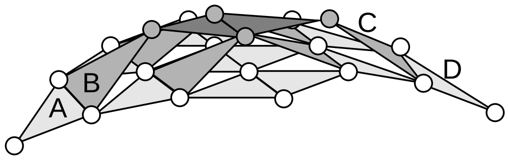

Fig. 10. C^3 conditions

$$\begin{aligned}\varpi G(V_1^{<i>}, V_2^{<j>}, D_0, D_0, V_0) &= \varpi F(V_1^{<i>}, V_2^{<j>}, D_0, D_0, V_0) \\ &= d_0 \varpi F(V_1^{<i>}, V_2^{<j>}, D_0, V_0, V_0) + \\ &\quad d_1 \varpi F(V_1^{<i>}, V_2^{<j>}, D_0, V_0, V_1) + \\ &\quad d_2 \varpi F(V_1^{<i>}, V_2^{<j>}, D_0, V_0, V_2), \end{aligned} \tag{5}$$

where $i + j = n - 3$, $i, j \geq 0$.

Figure 10 illustrates C^3 condition. In this figure, the white points (connected with the light gray panels) are the control points of our two patches. The medium gray panels extend from the control points to give the gray points (some of which were constructed as part of the C^2 conditions), and the dark gray panel illustrates the coplanarity constraint given in Equation 5.

To test if two patches meet with C^3 continuity, consider each group of 16 control points along the boundary in the patch pair and first test the C^3 conditions as follows: extend panel A of Figure 10 to get the third point of panel B, and extend panel D to get the third point of panel C. For the patches to meet with C^3 continuity, the dark gray panels must be coplanar.

If the patches meet with C^2 continuity but not C^3 continuity, then we can adjust to achieve C^3 continuity in the following manner: Extend panels A and D to build the third points of panels B and C. We can now use the adjustment schemed used for C^1 continuity (Figure 6, Equation 2) to make the dark gray panels coplanar. This may cause panels A and B to no longer be coplanar; in this case, extend the modified panel B to get a new position for the third point of panel A. A similar adjustment is made for panels C and D.

Again we see that if the two patches originally meet with C^3 continuity, then none of the control points of the two patches will be changed by this construction.

2.5 Continuity Conditions as an Averaging of Polynomials

Mike Floater [3] has pointed out that this averaging of control points construction can be viewed as averaging the two polynomials. If p_1 is the initial polynomial patch constructed for one triangle, and p_2 is the initial polynomial patch constructed for the adjacent triangle, then $P = (p_1 + p_2)/2$ is the aver-

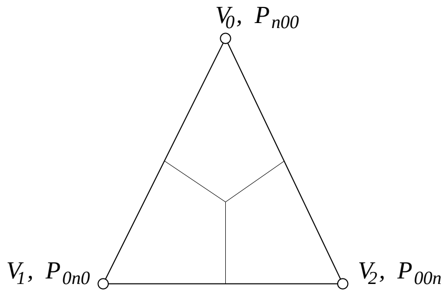

Fig. 11. Control points in each region are set by data at the closest V_i. Control points between regions have averaged values.

age of these two polynomials. What the above averaging schemes do is select layers of control points from P until the desired continuity is met. Since both sides are selecting their control points from the same P, the two patches meet with the desired continuity.

Note that with Floater's view of the continuity control point adjustments, we still need a process similar to the ones described above to construct the control points of P, since at least one of p_1 or p_2 will need to be reparameterized over the other's domain.

3 A Simple Example

As an example of using my continuity adjustment scheme, I present a simple interpolant that first constructs a C^0, degree n patch network that sets all the control points by interpolating derivatives at the data points. This patch network will reproduce degree n polynomials. Next, I use the continuity adjustment scheme to increase the continuity without losing polynomial precision. This interpolant requires $\lfloor 2n/3 \rfloor$ complete derivatives at each data points, although I will not use all of the higher order derivatives. The result will be a $C^{\lfloor (n-1)/4 \rfloor}$ interpolant that reproduces degree n polynomials.

More precisely, for P_{ijk}, with $i > j$, $i > k$, I use the data at V_0 to set the value of P_{ijk}. If $i < j = k$, then I compute two values using the data at V_1 and V_2 and average them. If $i = j = k$, then I compute three values, using the data at V_0, V_1, and V_2 and average the result. The remaining cases are handled in a similar fashion. This divides the data triangle into three regions as illustrated in Figure 11.

With this scheme, when constructing a degree n patch, if the data at all three corners comes from a single degree n polynomial, then this scheme reproduces that polynomial. In cases where a control vertex is the average of two or three values computed from the corner data, these values will be identical, since the

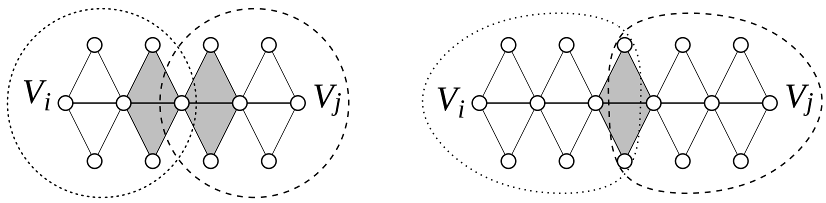

Fig. 12. Join is not C^1 because shaded panels not coplanar.

corner data come from a single polynomial.

The piecewise interpolant filling a triangular network will create a C^0 piecewise polynomial surface. The C^0 continuity conditions are met because the boundary control points between two adjacent patches are computed using the same data. However, in general the patches will not meet with C^1 continuity, since in the case when n is even, the middle boundary point is an average of two values (Figure 12), and the panels adjoining this boundary point will not be coplanar; when n is odd, a similar problem occurs for the cross-boundary points.

3.1 Increasing Continuity

Two neighboring patches built with this construction will only meet with C^0 continuity. However, only one or two panels along the boundary are out of C^1 alignment, as illustrated in Figure 12. Thus, if we apply the C^1 adjustment to these panels (a single panel when the degree is odd, two panels when the degree is even), then the two patches will meet with C^1 continuity.

We must be careful, however, that the vertices adjusted for one boundary do not affect the C^1 connection along another boundary. In particular, degree 5 is the lowest degree for which we can use this C^1 adjustment on a patch network [10]. That we can use degree 5 patches is illustrated in Figure 13; here, the shaded vertices are the ones we need to adjust get C^1 continuity, and each shaded vertex only affects the C^1 continuity across a single boundary.

In general, while we can apply the averaging construction given in this paper to pairs of patches, to apply it to a network of patches the degree of patch required by my constructions is $4k + 1$, where k is the level of continuity desired. This requirement is needed so that the vertices adjusted to achieve C^k continuity along one boundary are not involved in the C^k conditions along another boundary. This $4k+1$ condition agrees with the result of Ženíšek [10].

This interpolant requires a large number of derivatives at the data points. I have also looked at two variations on this scheme that require fewer derivatives, and at Clough-Tocher schemes that can obtain higher continuity with lower

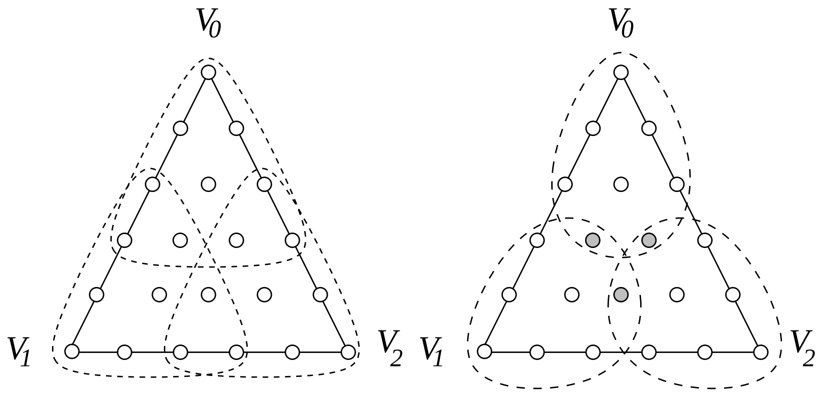

Fig. 13. Derivatives needed for the quintic version of simple data fitting scheme.

degree, as described in two technical reports [7,8]. The second of these reports also discusses testing of convergence properties and looks at the shape of the constructed surfaces.

4 Summary

This paper has presented a method for increasing the continuity of triangular Bézier patches. The adjustment scheme has the property that if two patches already meet with the desired continuity, then relevant control points remain unchanged. I then gave an example showing how to use this adjustment scheme to build a degree n interpolant that reproduces degree n polynomials, with the patches meeting with the highest possible continuity.

As a final note, while the averaging construction for obtaining continuity has only been applied here for triangular Bézier patches, the idea is easily extended to curves, and in turn, to tensor product Bézier surfaces. In the latter case, we would want (as we did for triangles) enough derivatives at the data points to consistently set enough mixed partial derivatives for the continuity adjustments we are applying to the patches.

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