Counting Pruned Bézier Curves

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Abstract. Removing some intermediate computations from the de Casteljau algorithm for Bézier curves generates a new class of curves called pruned Bézier curves. This class of curves includes the classical Ball curves and their generalizations as well as curves evaluated by Horner's method. Here, we study the combinatorics of pruned Bézier curves. By solving recurrence relations we derive closed formulas for the number of pruned curves of various types, including those that are non-degenerate (bushes), those that are variation diminishing (hedges), those whose algorithms are symmetric, and those whose algorithms are distinct. We also observe that combinatorially these non-degenerate pruned Bézier schemes (bushes) are related to binary and ternary trees.

§1. Introduction

The de Casteljau algorithm for evaluating Bézier curves consists of repeated affine combinations that can be arranged in a triangular fashion [6]. Trimming some of the nodes from this triangular diagram gives the computation for Ball curves, for curves in monomial form, and more generally for several classes of polynomial curves known as pruned Bézier curves.

In earlier work [2], the geometry of two classes of pruned Bézier curves — bushes and hedges — were investigated. Bushes are non-degenerate curves; their basis functions are linearly independent. Hedges are variation diminishing curves; their basis functions are totally positive. Both bushes and hedges are affine invariant and lie in the convex hull of their control points.

The pruning rules for bushes are that a node x in the triangle diagram may be removed if

- x is a leaf node
- x has exactly one parent

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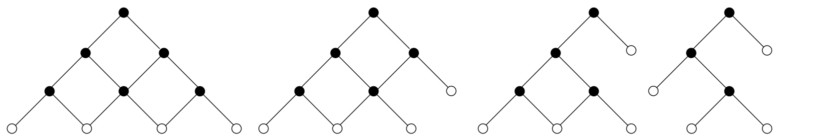


Fig. 1. Illustration of pruning process for bushes. Dark nodes indicate computations; hollow nodes indicate control points.

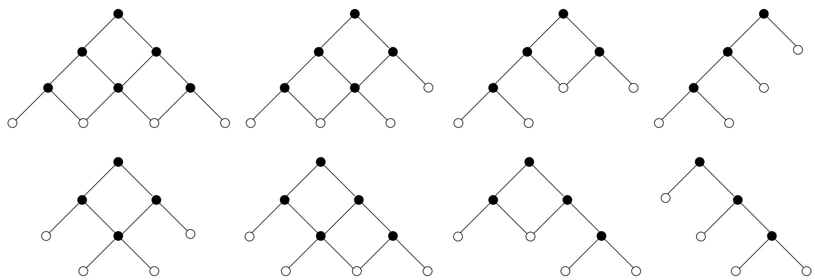


Fig. 2. All degree 3 hedges.

- x has a sibling with exactly two parents

An example of the pruning process for depth 3 bushes appears in Figure 1. The pruning rules for hedges are the same as for bushes, with one additional restriction on when a node x may be pruned:

- x's sibling is also a leaf

All depth 3 hedges are shown in Figure 2.

A key distinction between the pruning rules for bushes and hedges is the following. For hedges, if we delete $k > 0$ nodes from one side of the jth level, we must delete at least $k + 1$ nodes from the same side of the $j + 1$st level, whereas for bushes, we need only delete k nodes from the $j + 1$st level.

In this paper, we will count the number of bushes of depth n and the number of hedges of depth n. In addition, we will consider restricted classes of bushes and hedges, and count the number of members in these classes.

§2. Counting Hedges

We start by counting the number of *right hedges*. In a right hedge, we are allowed to remove nodes only from the right side of the hedge. Left

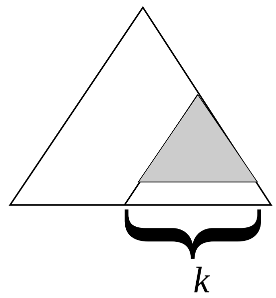


Fig. 3. Removing k nodes from bottom layer.

hedges can be defined in a similar fashion. In a right hedge with k nodes removed from the bottom layer, the only other nodes that may be removed form a triangle above those removed from the bottom layer (Figure 3).

Lemma 1. *Let H_k^n denote the number of right hedges we can generate if we remove at most k nodes from the right side of the nth level, with $k < n$. Then $H_k^n = 2^k$.*

Proof: By induction on k. Base case: $k = 0$. In this case, no nodes are removed, so $H_0^n = 1$.

Assume that $H_i^n = 2^i$ for $0 \leq i \leq k$.

Let EH_k^n denote the number of hedges with exactly k nodes removed from the right side of the nth level. To count the number of right hedges, H_{k+1}^n, we can form when removing up to $k+1$ nodes from the bottom row, we will sum EH_i^n, for $i = 0$ to $k+1$.

The number of right hedges we can form when we remove exactly i nodes from the bottom layer is equal to the number of right hedges we can form when we remove up to $i-1$ nodes on the next layer, (see the shaded region in Figure 3). Thus, for $i > 0$

$$EH_i^n = H_{i-1}^{n-1},$$

and $EH_0^n = 1$. Invoking the inductive hypothesis yields

$$H_{k+1}^n = \sum_{i=0}^{k+1} EH_i^n = 1 + \sum_{i=1}^{k+1} H_{i-1}^{n-1} = 1 + \sum_{i=1}^{k+1} 2^{i-1} = 2^{k+1}.$$

□

Note that the number of left hedges of depth n that can be generated by removing at most k nodes from the nth level is equal to H_k^n.

We are now ready to count the total number of hedges of depth n.

Theorem 1. *Let H^n denote the number of hedges of depth n. Then $H^n = (n+1)2^{n-2}$.*

Proof: Hedges of depth n are generated by removing nodes from the left side and nodes from the right side. As long as no more than $n-1$ nodes in total are removed from the bottom layer, removing additional nodes at higher levels in the hedge can be done independently on the left and right sides.

To count the total number of hedges we can form when we remove exactly i nodes from the bottom left, we multiply the number of left hedges we can generate when removing exactly i nodes from the bottom level by the number of right hedges that result from removing up to $n-1-i$ nodes from the bottom level.

Thus, the number of hedges we can form when we remove exactly i nodes from the left (with $n > i > 0$) is $EH_i^n \times H_{n-i-1}^n = 2^{i-1} \times 2^{n-1-i} = 2^{n-2}$. We compute H^n by counting the number of hedges we can generate when we remove exactly i nodes on the left, and summing over i:

$$\begin{aligned} H^n &= \sum_{i=0}^{n-1} EH_i^n \cdot H_{n-i-1}^n \\ &= H_{n-1}^n + \sum_{i=1}^{n-1} H_{i-1}^{n-1} \cdot H_{n-i-1}^n \\ &= 2^{n-1} + \sum_{i=1}^{n-1} 2^{i-1} 2^{n-i-1} \\ &= 2^{n-1} + (n-1)2^{n-2} = (n+1)2^{n-2} \end{aligned}$$

□

2.1. The Number of Symmetric Hedges

The number of symmetric hedges of depth n (SH^n) is equal to the number of right hedges where the nodes removed do not extend past the middle. Counting even and odd n separately gives us

$$\begin{aligned} SH^{2n} &= H_{n-1}^{2n} = 2^{n-1} \\ SH^{2n+1} &= H_n^{2n+1} = 2^n \end{aligned}$$

and we see that $SH^n = 2^{\lceil (n-2)/2 \rceil}$.

2.2. The Number of Distinct Hedges

When we calculated H^n, we counted hedges with the same symmetry twice. The number of distinct hedges of depth n (DH^n, where we count symmetric hedges only once) will be half the sum of the total number of hedges and the number of symmetric hedges. Again counting even and odd n separately gives us

$$\begin{aligned} DH^{2n} &= \frac{SH^{2n}+H^{2n}}{2} = 2^{n-2} + (2n+1)2^{2n-3} \\ DH^{2n+1} &= \frac{SH^{2n+1}+H^{2n+1}}{2} = 2^{n-1} + (2n+2)2^{2n-2} \end{aligned}$$

and we see that

$$DH^n = 2^{\lceil (n-1)/2 \rceil -1} + (n+1)2^{n-3}.$$

§3. Counting Bushes

The method we used for counting hedges (i.e., counting left and right hedges independently) will not work for bushes, since for bushes once we remove nodes from the left and right on the bottom layer, the nodes we can remove at higher layers are not independent. Instead, to count the number of bushes of depth n, we begin by counting the number of symmetric bushes. We will show that the number of symmetric bushes of depth $2n$ is equal to the number of bushes of depth n.

3.1. Symmetric Bushes

Let SB^n denote the number of symmetric bushes of depth n, let SB_k^n denote the number of depth n symmetric bushes with at most $2k$ nodes removed from the nth level and let ESB_k^n denote the number of depth n symmetric bushes with exactly $2k$ nodes removed from the nth level. The key observation in counting the number of symmetric bushes is

$$ESB_k^n = SB_k^{n-1}.$$

This result is illustrated in the left diagram of Figure 4; if we remove exactly k nodes from each side of the bottom level of a depth n symmetric bush (the light gray region in the figure), then we can form symmetric bushes by removing nodes in the dark gray regions of the figure, symmetrically across the vertical line. For this equality to hold in all cases, we must make the following special definition:

$$SB_n^{2n} = SB_{n-1}^{2n}. \tag{1}$$

The need for this boundary condition is seen in the right diagram of Figure 4, where having removed two nodes on either side of the bottom level, we are unable to remove two nodes on either side of the next higher layer, since that would remove node D from the bush.

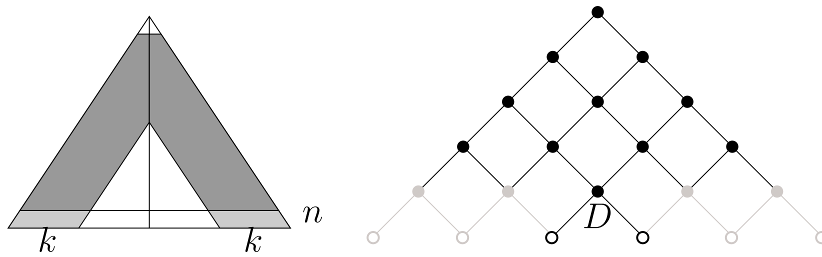


Fig. 4. Illustration of $ESB_k^n = SB_k^{n-1}$.

To compute the number of symmetric bushes, we note that SB_k^n (the number of symmetric bushes of depth n with at most k nodes removed from each side on the nth level) is equal to the number of symmetric bushes of depth n with exactly k nodes removed from each side plus the number of symmetric bushes of depth n with no more than $k-1$ nodes removed from each side. This observation gives us

$$\begin{aligned} SB_k^n &= ESB_k^n + SB_{k-1}^n \\ &= SB_k^{n-1} + SB_{k-1}^n, \end{aligned} \tag{2}$$

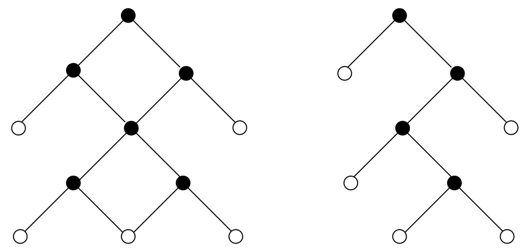
when $1 \le k \le \lceil (n-1)/2 \rceil$. When $k = 0$, we note that $SB_0^n = 1$. The closed form solution of this recurrence (with boundary condition (1)) can be found using standard combinatorial techniques (see for example [1]):

$$SB_k^n = \binom{n+k-1}{k} - \binom{n+k-1}{k-1} - 2\binom{n+k-1}{k-2}. \tag{3}$$

Note that the correctness of the closed form can by checked by verifying that (3) satisfies (2) and (1).

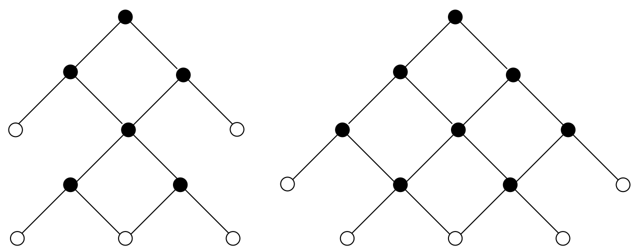
To count all bushes, we will need SB^{2n}, so we note that, with a bit of simplification, we have the following:

$$\begin{aligned} SB^{2n} &= SB_{n-1}^{2n} \\ &= \binom{3n-2}{n-1} - \binom{3n-2}{n-2} - 2\binom{3n-2}{n-3} \\ &= \frac{\binom{3n}{2n+1}}{n} \\ SB^{2n+1} &= SB_n^{2n+1} \\ &= \frac{\binom{3n+1}{2n+1}}{n+1} \end{aligned}$$



(0,0), (0,0), (1,1), (1,1)   (0,0), (1,0), (1,1), (2,1)

Fig. 5. Bushes as sequences of pairs of integers.



0,0,1,1   0,0,0,1

Fig. 6. Symmetric bushes as sequences of integer.

3.2. Characteristic Sequences and Bushes

Let a_k denote the number of nodes removed from the kth level of a bush from the left; let b_k denote the number of nodes removed from the kth level on the right; and let c_k denote the number of nodes removed from the kth level of each side of a symmetric bush.

We can now describe a bush of depth n abstractly as a sequence of pairs of integers, $(a_1, b_1), \ldots, (a_n, b_n)$ with $0 = a_1 \leq a_2 \leq \cdots \leq a_n$ and $0 = b_1 \leq b_2 \leq \cdots \leq b_n$, and $0 \leq a_k + b_k \leq k - 1$. Figure 5 gives two examples of bushes and their corresponding sequences of integer pairs. We can also describe symmetric bushes of depth $2n$ abstractly as sequences of integers, $c_1, \ldots, c_{2n}$ where $0 = c_1 \leq \cdots \leq c_{2n}$, with $0 \leq c_{2k-1} \leq k - 1$ and $0 \leq c_{2k} \leq k - 1$. Figure 6 shows two examples of symmetric bushes and their corresponding sequences of integers.

Theorem 2. *There is a bijection between the number of bushes of depth n and the number of symmetric bushes of depth $2n$.*

Proof: We know $c_1 = 0$. For all other c_i, the following gives a mapping of the sequences of integer pairs corresponding to a bush of depth n to the integer sequence corresponding to a symmetric bush of depth n:

$$\begin{aligned} c_{2k-1} &= a_k + b_{k-1} \\ c_{2k} &= a_k + b_k \end{aligned}$$

Reversing the process, starting with $a_1 = b_1 = 0$, we find that for a_i, b_i, with $i > 1$ we have

$$\begin{aligned} a_k &= c_{2k-1} - b_{k-1} \\ b_k &= c_{2k} - a_k. \end{aligned}$$

These mappings of sequences establish a bijection between bushes of depth n and symmetric bushes of depth $2n$. □

Corollary 1. *Let B^n denote the number of bushes of depth n. Then*

$$B^n = \frac{\binom{3n}{2n+1}}{n}.$$

3.3. Distinct Bushes

Let $DB(n)$ denote the number of distinct bushes of depth n. The number of distinct bushes is half the sum of the number of bushes of depth n and the number of symmetric bushes of depth n. Counting even and odd n separately, we have

$$\begin{aligned} DB(2n) &= \frac{SB^{2n} + B^{2n}}{2} = \frac{SB^{2n} + SB^{4n}}{2} \\ &= \frac{2\binom{3n}{2n+1} + \binom{6n}{4n+1}}{4n} \\ DB(2n+1) &= \frac{SB^{2n+1} + B^{2n+1}}{2} = \frac{SB^{2n+1} + SB^{4n+2}}{2} \\ &= \frac{\binom{3n+1}{2n+1}}{2n+2} + \frac{\binom{6n+3}{4n+3}}{4n+2} \end{aligned}$$

3.4. Right bushes

We shall now count the number of right bushes of depth n and show that this count is equal to the number of binary trees with n nodes.

Let B_k^n denote the number of bushes with at most k nodes removed from the right side of the nth level and let EB_k^n denote the number of bushes with exactly k nodes removed from the right side of the nth level. As with symmetric bushes, the key observation in counting the number of right bushes is that

$$EB_k^n = B_k^{n-1},$$

but with the boundary condition

$$B_n^n = B_{n-1}^n. \tag{4}$$

We now establish a recurrence relation for right bushes by observing that the number of depth n right bushes with at most k nodes removed from the nth level is equal to the number of right bushes where we remove exactly k nodes from the right plus the number of right bushes of depth n where we remove at most $k-1$ nodes from the right. Thus, for $1 \le k \le n$,

$$\begin{aligned} B_k^n &= EB_k^n + B_{k-1}^n \\ &= B_k^{n-1} + B_{k-1}^n, \end{aligned}$$

and $B_0^n = 1$.

Solving for a closed form solution of this recurrence with boundary condition 4 in a manner similar to that used for symmetric bushes gives us

$$B_k^n = \binom{n+k}{k} - \binom{n+k}{k-1} = \frac{(n+1-k)(n+k)!}{k!(n+1)!}.$$

The number of binary and ternary trees with n nodes is well known [4, 7]. From our explicit formulas, we observe the following two relationships between right bushes and binary trees, and between bushes and ternary trees.

Theorem 3. $$\begin{aligned} B_{n-1}^n &= \text{the number of binary trees with } n \text{ nodes} \\ &= \frac{(n+1-k)(n+k)!}{k!(n+1)!}. \end{aligned}$$

Theorem 4. $$\begin{aligned} B^n &= \text{the number of ternary trees with } n \text{ nodes} \\ &= \frac{\binom{3n}{2n+1}}{n} \end{aligned}$$

§4. Open Questions and Future Work

We have derived closed formulas both for the number of hedges and for the number of bushes of a fixed depth. Suppose, however, that instead of fixing the depth, we fix the number of nodes – or equivalently the number of processors [5] – in the evaluation algorithm. We would like to know how many hedges or bushes we can construct if we are limited to a fixed number of processors. Preliminary investigations lead us to believe that answers to these types of questions are related to the theory of integer partitions – that is, to the representations of positive integers by sums of other positive integers. Closed formulas that count different kinds of integer partitions are typically hard to come by, but recurrence relations and even generating functions are often available. We plan to study this topic further and report on our results in a future paper.

§5. Acknowledgements

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§6. References

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